Approximation Algorithms

Assignment 3

Due on : 13.4.2018

Please write solutions independent of each other. You are welcome and in fact encouraged to discuss, but strictly prohibited from copying solutions from another person. No external sources other than class notes/lecture notes uploaded may be consulted for solving these problems. Violation of this would be considered an act of plagiarism

Problem 1

a) (3 points) There are n jobs and m machines ; each machine has a capacity of 1. The processing time for job j on machine i is p_{ij} and the profit of this assignment is c_{ij}. Following is an LP relaxation to the following problem : Assign jobs to machines in order to maximize the total profit subject to the constraint that the total processing time on each machine is at the most its capacity, i.e, 1.

$$\begin{aligned} \max \sum_{i,j} c_{ij} x_{ij} \\ \text{s.t.: } \sum_{j} x_{ij} p_{ij} \leq 1, \forall i \\ \sum_{i} x_{ij} \leq 1, \forall j \\ x \geq 0 \end{aligned}$$

Write the dual to the above LP.

b) (4 points) Given an undirected graph $G = (V, E)$ and non-negative capacities $c(u, v)$ for all edge $(u, v) \in E$. We take *distance* variables $d(u, v)$ for each pair of vertices $(u, v) \in V$. Let s and t be a source and sink vertex respectively. Now consider the following LP relaxation :

$$\begin{aligned} \min \sum_{(u,v) \in E} c(u, v) d(u, v) \\ \text{s.t.: } d(u, v) \leq d(u, w) + d(w, v), \forall (u, v, w \in V) \\ d(s, t) = 1 \\ d \geq 0 \end{aligned}$$

Which problem do you think the above LP is a relaxation of and why? Write the dual of the LP. Does the dual form a relaxation to any problem you know about?

Problem 2 Given an undirected weighted graph $G(V, E)$, the following is known as the spanning tree polytope. Let $x(E[S]) = \sum_{e=(u,v):\{u,v\} \subseteq S} x_e$.

$$\mathcal{P} = \{x \in \mathbb{R}^{|E|} : \sum_{e \in E} x_e = |V| - 1, x(E[S]) \leq |S| - 1, \forall S \subset V\}$$

Design a polynomial time separation oracle for the problem, i.e., given any solution x, the oracle should either output that x is a point in the polytope $\mathcal{P}$ or output a constraint that is violated. (Hint : Try to design a flow network with suitable capacities. Then show that the separation oracle is essentially solving several min-cut problems in the network.).

Problem 3 (Bonus Problem) Prove that the following randomized algorithm for set cover is an $\mathcal{O}(f)$-approximation in expectation. Consider the uncovered elements in any order. For an uncovered element e, pick the set $S : e \in S$ with probability $\frac{1/c(S)}{\sum_{S' : e \in S'} 1/c(S')}$.