Iterated Rounding : Generalized Steiner Network

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I/P : Undirected Graph $G = (V,E)$; $c : E \to R_{\geq 0}$

$D = \{\{s_i, t_i\} \subseteq V : i = 1, 2, \ldots, k\}$; source-sink pairs

$r : D \to \mathbb{Z}_{\geq 0}$; requirement of pair $\{s_i, t_i\}, i = 1, 2, \ldots, k$

O/P : A subgraph $F \subseteq G$: F contains r_i edge-disjoint paths between s_i & t_i.

Obj : Minimize $c(F)$.

LP-SNDP:

$$\min \sum x_e c_e$$

$$\text{s.t.} \quad \sum_{e \in \partial(S)} x_e \geq f(S), \ \forall S \subset V$$

$$1 \geq \bar{x} \geq 0$$

where, $f(S) = \min_{i : |\{s_i, t_i\} \cap S| = 1} r_i$: If S is an $s_i - t_i$ cut, then there should be r_i edges in $\partial(S)$ (Menger's Thm)

Amazing Fact: At any extreme point solution of LP-SNDP, if $x_e > 0 \ \forall e \in E$, then $\exists e \in E$, s.t. $x_e \geq 1/2$.

Given this fact (which we prove later), we run the following iterative algorithm.

Alg-SNDP: Initialize: $E' \leftarrow E$; $F \leftarrow \emptyset$; $f' \leftarrow f \ \forall S$

While $f'(S > 0$ for some S,

Solve LP-SNDP on residual instance $G' = (V, E')$ & f'.

i) If $x_e = 0$, $E' \leftarrow E' - \{e\}$

Solve LP-SNDP on residual instance $I = (V, E'), f'$.

i) If $x_e = 0$, $E' \leftarrow E' - \{e\}$

ii) If $x_e \geq 1/2$, $E' \leftarrow E' - \{e\}$, $F \leftarrow F \cup \{e\}$
$f'(S) \leftarrow f'(S) - 1$ if $e \in \partial(S)$.

Return F.

Lemma 1. F is a 2-approximate solution to SNDP on G, f.

Proof: Convince yourself that F is feasible.

Now, we use induction to prove that F is 2-approximate. In fact, we show the stronger claim that

$c(F) \leq 2 \sum_{e \in E} x_e^* c_e$, x_e^*: OPT solution to LP-SNDP.

We use induction on the number of iterations. Let the above be true if the algorithm above has t iterations.
(Convince yourself about the base case, $t = 1$)

Now, assume the algorithm runs for $t+1$ iterations. Let e' be the edge that was ~~disc~~ discarded or selected at the first iteration. Let $F' = F - \{e'\}$, i.e, the set of edges selected in the subsequent t iterations. Let $x_{(res)}$ be an optimal solution to the residual LP-SNDP at iteration 2. Hence, by induction hypothesis,

$$c(F') \leq 2 \sum_{e \in E - \{e'\}} x_{res} \cdot c_e$$

Now we make the crucial observation that x_e^*, restricted to the components $E - \{e'\}$ is a feasible solution to x_{res}. This can be easily verified by checking all constraints for a cut $S \subseteq V$. This gives us

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$$c(F) \leq c(F') + c_{e'} \leq 2\sum_{e\in E-\{e'\}} x_{res}\cdot c_e + c_{e'} \overset{\text{above observation}}{\leq} 2\sum_{e\in E-\{e\}} x^*_e c_e + c_{e'}$$

$$\underset{(x^*_{e'}\geq 1/2)}{\leq} 2\sum_{e\in E} x^*_e c_e + 2x^*_{e'} c_{e'}$$

$$\leq 2\cdot\sum_{e\in E} x^*_e c_e \quad \blacksquare$$

The above Lemma, in fact, gives an upper bound on the integrality gap of LP-SNDP, even though the algorithm solves a different LP everytime. In the original paper by Kamal Jain (circa '98), he, in fact, shows how to avoid solving the LP iteratively. The idea is that, one can solve the LP once and then "jump" to an extreme point of the residual LP.

Now we prove the Amazing fact. For this, we need two crucial properties as follows.

- **Property I**: Let $x(E') = \sum_{e\in E'} x_e$. Then the function

  $x(\partial(S))$ is $\boxed{\text{strongly submodular}}$.

  Proof: We need to show the following. Let S_1 and S_2 be two cuts in G. Then

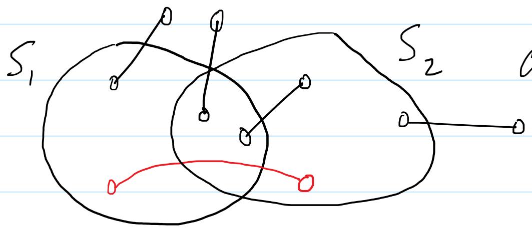
  $$\text{i)}\ x(\partial(S_1)) + x(\partial(S_2)) \geq x(\partial(S_1\cup S_2)) + x(\partial(S_1\cap S_2))$$

  and

  $$\text{ii)}\ x(\partial(S_1)) + x(\partial(S_2)) \geq x(\partial(S_1\setminus S_2)) + x(\partial(S_2\setminus S_1))$$

We show i and leave ii for you to verify.

Define $\chi(E') \in \{0,1\}^{|E|}$ to be a



Define $\chi(E') \in \{0,1\}^{|E|}$ to be a characteristic vector of the subset $E' \subseteq E$ of E such that

$$\chi(E',e) = 1 \text{ if } e \in E'$$
$$= 0 \text{ o.w.}$$

Then we have
$$\chi(\partial(S_1 \cup S_2)) = \chi(\partial(S_1)) + \chi(\partial(S_2)) - \chi(\partial(S_1 \cap S_2)) \underbrace{- \chi(E(S_1,S_2))}_{\text{All edges between } S_1 \,\&\, S_2},$$

Multiplying throughout by the vector x,
$$x(\partial(S_1 \cup S_2)) = x(\partial(S_1)) + x(\partial(S_2)) - x(\partial(S_1 \cap S_2)) - x(E(S_1,S_2))$$
$\Rightarrow$ (i) (Changing sides)

Note that strict equality holds iff. $x(E(S_1,S_2)) = 0$
Hence, if $x > 0$, then strict equality holds iff. $\chi(E(S_1,S_2)) = 0$.

$\blacksquare$

<u>Property II</u>. The function $f(S)$ is weakly supermodular, i.e.,

either i) $f(S_1) + f(S_2) \leq f(S_1 \cup S_2) + f(S_1 \cap S_2)$

or ii) $f(S_1) + f(S_2) \leq f(S_1 - S_2) + f(S_2 - S_1)$

<u>Proof</u>: Exercise.

<u>Lemma 2</u>. If S_1, S_2 are both tight sets at some solution x, then so are $S_1 \cup S_2$ & $S_1 \cap S_2$ OR $S_1 - S_2$ & $S_2 - S_1$.

<u>Proof</u>: Since S_1, S_2 are tight at x,

$$f(S_1) + f(S_2) = x(\partial(S_1)) + x(\partial(S_2)) \underbrace{\geq}_{\text{Property I, i}} x(\partial(S_1 \cup S_2)) + x(\partial(S_1 \cap S_2))$$

$$\underbrace{\geq}_{\text{Feasibility of } x} f(S_1 \cup S_2) + f(S_1 \cap S_2)$$

$$\geq f(S_1) + f(S_2) \text{ [Assuming Property II, i]}$$

Hence, all inequalities hold with strict equalities.

$$\Rightarrow x(\partial(S_1 \cup S_2)) + x(\partial(S_1 \cap S_2)) = f(S_1 \cup S_2) + f(S_1 \cap S_2)$$

$$\Rightarrow x(\partial(S_1 \cup S_2)) = f(S_1 \cup S_2) \; \& \; x(\partial(S_1 \cap S_2)) = f(S_1 \cap S_2)$$

Note that in the case when Properl II, ii holds, we can use a very similar argument to show

$x(\partial(S_1 - S_2))$ & $x(\partial(S_2 - S_1))$ are tight.

<u>Corollary</u>: If $x > 0$, then S_1, S_2 are tight

$\Rightarrow$ either a) $\chi(S_1) + \chi(S_2) = \chi(S_1 \cup S_2) + \chi(S_1 \cap S_2)$

or b) $\chi(S_1) + \chi(S_2) = \chi(S_1 - S_2) + \chi(S_2 - S_1)$

Lemma 3. Given an extreme pt. x of LP-SNDP such that $1 > x_e > 0 \;\; \forall e \in E$, there exists a set $\mathcal{L}$ of TIGHT constraints at x, s.t.

i) $\mathcal{L}$ is linearly independent
ii) $|\mathcal{L}| = |E|$
iii) $\mathcal{L}$ is a laminar family.

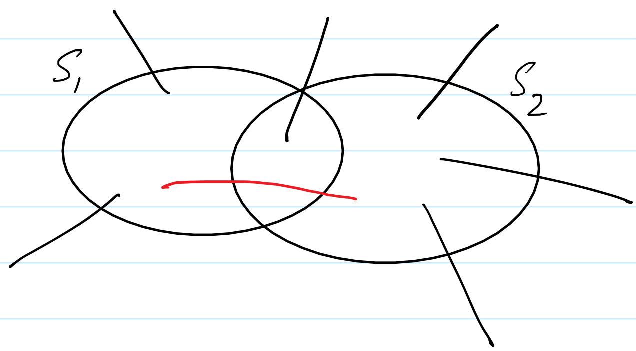
<u>Proof</u>: Let $T = \{ S : x(\partial(S)) = f(S) \}$, i.e., all tight cuts at x. Rank Lemma gives us that since x is an extreme pt., it must be defined by $|E|$ linearly independent tight constraints. Further, these tight constraints can only be the cut-constraints ($\because 1 < x_e < 0$). Hence, $\mathcal{L} \subseteq T$ and $|\mathcal{L}| = |E|$. Next we prove iii using a separate lemma which is also called the Uncrossing Lemma. ▨

<u>Lemma 4</u> <u>(Uncrossing Lemma)</u>: $\mathcal{L}$ is a laminar family.

Lemma 4 (Uncrossing Lemma): $\mathcal{L}$ is a laminar family.

Proof: The proof is constructive. We start with a set $\mathcal{L}$ that is non-laminar and convert it to a laminar family. One has to be careful not to violate i) and ii) of Lemma 3 in the process.

Assume $\mathcal{L}$ is non-laminar $\Rightarrow \exists S_1$ and S_2 such that all sets $\underbrace{S_1 - S_2,\ S_1 \cap S_2\ \&\ S_2 - S_1 \text{ are non-empty.}}_{\Rightarrow S_1 \,\&\, S_2 \text{ are pair of ``Crossing Sets''}}$



Since S_1, S_2 are tight, Lemma 2 $\Rightarrow S_1 \cup S_2$ & $S_1 \cap S_2$ are also tight. i.e, $S_1 \cup S_2 \in T$, $S_1 \cap S_2 \in T$. (Either this or $S_1 - S_2$ & $S_2 - S_1 \in T$. Let's assume the first case.)

Further, since $x > 0 \Rightarrow x(\partial(S_1)) + x(\partial(S_2))$
$$= x(\partial(S_1 \cup S_2)) + x(\partial(S_1 \cap S_2))$$

But this means $S_1, S_2, S_1 \cup S_2, S_1 \cap S_2$ are not linearly independent $\Rightarrow$ at least one of $S_1 \cup S_2$ and $S_1 \cap S_2$ belongs to $T - \mathcal{L}$

Assume the first one is true. Then update
$$\mathcal{L} \leftarrow \mathcal{L} - \{S_1\} \cup \{S_1 \cup S_2\}$$

(If both $S_1 \cup S_2$, $S_1 \cap S_2$ are in $T - \mathcal{L}$, then $\mathcal{L} \leftarrow \mathcal{L} - \{S_1, S_2\} \cup \{S_1 \cup S_2, S_1 \cap S_2\}$)

[you may remove S_2 as well instead of S_1]

Let's now verify that we did not violate i) or ii). (ii) is simple since we add a set and remove exactly one from $\mathcal{L}$. We can prove (ii) showing that we did not introduce any linear dependency in the above update. Assume for contradiction that we actually introduced linear dependency among some cuts $S_1', S_2', \ldots, S_k' \in \mathcal{L}$. However, one of S_i' must be the newly introduced set $S_1 \cup S_2$ since otherwise $\mathcal{L}$ was not linearly independent before update.

$$\Rightarrow \chi(\partial(S_1')) + \chi(\partial(S_2')) + \cdots + \chi(\partial(S_k')) = 0$$

$$\Rightarrow \chi(\partial(S_1 \cup S_2)) + \sum_{i=2}^{k} \chi(\partial(S_i')) = 0$$

(assuming wlog $S_1' = S_1 \cup S_2$)

$$\Rightarrow \underbrace{\chi(\partial(S_1)) + \chi(\partial(S_2)) - \chi(\partial(S_1 \cap S_2))}_{\chi(\partial(S_1)) + \chi(\partial(S_2)) = \chi(\partial(S_1 \cup S_2)) + \chi(\partial(S_1 \cap S_2))} + \sum_{i=2}^{k} \chi(\partial(S_i')) = 0$$

This is a contradiction since all the above sets in the last ~~is~~ equality belongs to $\mathcal{L}$ by induction hypothesis. (Note that the above proof ~~or~~ works only if one of $S_1 \cup S_2$ or $S_1 \cap S_2$ is in $T - \mathcal{L}$. The other case, i.e, $S_1 \cup S_2$ & $S_1 \cap S_2$ are both in $T - \mathcal{L}$ is slightly longer and we avoid that here).

Finally, we prove that the number of crossings in $\mathcal{L}$ decreases by at least one. This follows from the fact that we cannot introduce any new crossing since any new crossing has to form with $S_1 \cup S_2$, but all these crossings were already existing.

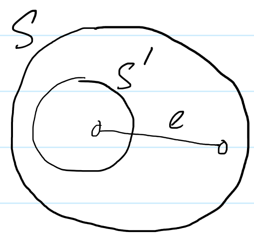
(A succint version of the proof exists in literature. However, I find them more technical an non-intuitive, although they are slick and elegant)

Finally, we are ready to prove the Amazing Fact using Lemma 3. We do this by a fractional charging argument.

Again, this is concise and less intuitive. One can find more intuitive and longer proofs in literature.

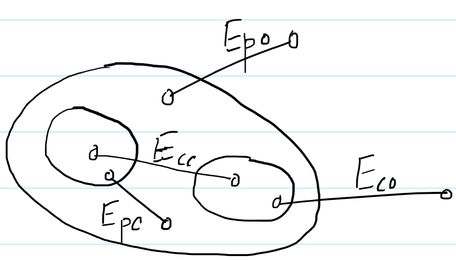
Proof (Amazing Fact). Assume for the sake of contradiction that $0 > x_e > 1/2$ $\forall e \in E$. $\Rightarrow$ both $1-2x_e$, $x_e > 0$.

We undertake a process where every edge $e \in E$ distributes some charge. Let $S \in \mathcal{L}$ be the minimal set that contains both end-points of e. Then e distributes $1-2x_e$ charge to S, if such a set exists.



Let S' be the minimal set such that one-endpoint of e belongs to S'. Then S' gets a charge of x_e units from e.

Now, any edge can distribute $1-2x_e+2x_e \le 1$ unit of charge in the above procedure. However, since $\mathcal{L}$ is a laminar family, then any maximal cut $S \in \mathcal{L}$ (a set such that $S' \subset S$ or $S' \cap S = \emptyset$ if $S' \in \mathcal{L}$) requires at least some edge $e' \in \partial(S)$. $\Rightarrow e'$ cannot distribute $1-2x_{e'}$ to any set and hence distributes at the most $2x_{e'} < 1$ unit of charge. Thus, total charge distributed is strictly less than $|E|$.



We prove now that any set $S \in \mathcal{L}$ gets at least 1 unit of charge. This, by Lemma 3, gives a contradiction since $|\mathcal{L}| = |E|$.

Consider any set $S \in \mathcal{L}$. In general, it contains subsets $C_1, C_2, \dots, C_k$ such that $C_i \in \mathcal{L}$, $i = 1, 2, \dots k$ and let C_i be the maximal subsets of S that belongs to $\mathcal{L}$. There can be 4 classes of edges as sho above:

$$E_{po} = \{ e : e \in \partial(S) \text{ but } e \notin \partial(C_i) \text{ for any } i=1,2,..k \}$$
$$E_{co} = \{ e : e \in \partial(C_i) \text{ for some } C_i \}$$
$$E_{cc} = \{ e : e \in E(C_i, C_j) \}$$
$$E_{pc} = \{ e : e \in E(S, C_i) \} - E_{cc}$$

We claim that $E_{po} \cup E_{cc} \cup E_{pc}$ is non-empty. Assume not.

$\Rightarrow \chi(\partial(S)) = \sum_{i=1}^{k} \chi(\partial(C_i))$ $\Rightarrow$ linear dependency
$\Rightarrow$ contradiction.

Hence, S receives charge of $\geq |E_{cc}| - 2x(E_{cc}) + x(E_{po}) + |E_{pc}| - x(E_{pc})$

Further,
$$x(\partial(S)) - x\Big(\bigcup_{i=1}^{k} \partial(C_i)\Big) = x(E_{po}) + \cancel{x(E_{co})} - 2x(E_{cc}) - \cancel{x(E_{co})} - x(E_{pc})$$
$$\| $$
$$f(S) - \sum_{i=1}^{k} f(C_i) = x(E_{po}) - 2x(E_{cc}) - x(E_{pc})$$
(since $S, C_i, i=1,..,k$ are tight)

Hence total charge received by $S \geq |E_{cc}| + |E_{pc}| - \big(f(S) - \sum_{i=1}^{k} f(C_i)\big)$
$\Rightarrow \xi$ is a positive integer (since charges are positive)
$\Rightarrow$ Charge received by $S \geq 1$. $\blacksquare$