Approximation Algorithms

End-Sem (Take Home)

Full Marks : 40

Deadline : April 30, 11 AM

Try to solve all the problems. Please write the solutions independent of each other. You are not allowed to discuss with anybody. No external sources other than class notes/lecture notes/book sections that have been uploaded may be consulted for solving these problems. Violation of any of the above would be considered an act of plagiarism. Please try to be brief and precise in your solutions. You may state any result mentioned/proved in the lectures without proof.

Problem 1 Recall the facility location problem. You are given a graph $G = (V, E)$ along with a partition of the vertex set in to F, the set of possible facility locations and C, the set of possible client locations. There is a facility opening cost f_i associated with opening a facility $i \in F$. Further, the cost of serving client $j \in C$ by facility $i \in F$ is d_{ij}. The objective is to open a set of facilities and assign all the clients to its nearest open facility so as to minimize the sum of total facility opening cost and total serving cost.

Design a greedy $\mathcal{O}(\ln n)$-approximation algorithm for the above problem.

(*Hint*: Try to imagine the optimal solution as a collection of 'stars', where each star has a facility at the center and clients attached to it. All you need to do is pick a bunch of stars so as to cover all clients. Have you seen something similar ? Can you adapt the greedy procedure for that problem for solving this one?)

Problem 3 Consider a primal-dual pair of linear programs as follows.

$$\min \sum_{i=1}^{n} c_i x_i \qquad\qquad \max \sum_{j=1}^{m} b_j y_j$$

$$\sum_{i=1}^{n} a_{ij} x_i \geq b_j, \forall j = 1, 2, \cdots m \qquad\qquad \sum_{j=1}^{m} a_{ij} y_j \leq c_i, \forall i = 1, 2, \cdots n$$

$$x \geq 0 \qquad\qquad y \geq 0$$

Given a pair of feasible solutions $(x^\star, y^\star)$, where $x^\star \in \mathbb{R}^n$ and $y^\star \in \mathbb{R}^m$, the complementary slackness conditions are as follows

i) (Primal conditions:) either $x_i^\star = 0$ or $\sum_{j=1}^{m} a_{ij} y_j^\star = c_i, \forall i = 1, 2, \cdots n$

ii) (Dual conditions:) either $y_j^\star = 0$ or $\sum_{i=1}^{n} a_{ij} x_i^\star = b_j, \forall j = 1, 2, \cdots m$

a) Prove that $(x^\star, y^\star)$ are optimal solutions if and only if the above conditions are satisfied.

b) Now write the natural LP relaxation for set cover. Prove that the algorithm that rounds every non-zero variable to 1 yields an f-approximation to set cover, where f is the maximum frequency of any element.(*Hint:* Use primal complementary slackness)

Problem 3 Given a directed graph $G = (V, A)$, with w_{ij} being the weight of the directed arc (i, j).

a) Prove that the following is an LP-relaxation to the *directed* max-cut problem

$$\begin{aligned} &\max \sum_{i,j} w_{ij} z_{ij} \\ \text{s.t. } & z_{ij} \le x_i, \forall i \in V, (ij) \in A \\ & z_{ij} \le 1 - x_j, \forall j \in V, (ij) \in A \\ & 0 \le x_i \le 1 \end{aligned}$$

b) Let U be one side of the cut. Consider the randomized algorithm that puts vertex $i \in V$ in U with probability $x_i/2 + 1/4$. Prove that this yields a 1/2-approximation to directed max-cut.

Problem 4 Given an undirected graph $G = (V, E)$, recall the spanning tree polytope defined by the following constraints as follows. Define $E[S] = \{(s, s') \in E : s \in S \text{ and } s' \in S\}$.

$$\begin{aligned} &\sum_{e \in E} x_e = |V| - 1 \\ &\sum_{e \in E[S]} x_e \le |S| - 1, \forall S \subset V, |S| \ge 2 \\ &x_e \ge 0 \end{aligned}$$

Prove that at any extreme point, if $x_e > 0, \forall e \in E$, then $x_e = 1, \forall e \in E$. You can use the following steps :

a) Let $\chi(F) \in \{0, 1\}^{|E|}$ be the characteristic vector of a subset $F \subseteq E$. Prove that

$$\chi(E[S]) + \chi(E[T]) \le \chi(E[S \cup T]) + \chi(E[S \cap T])$$

b) Use uncrossing and rank lemma to prove that there exists a *laminar* family of tight constraints, $\mathcal{L}$ such that

(i) $\mathcal{L}$ is laminar

(ii) $\mathcal{L}$ is linearly independent

(iii) $|\mathcal{L}| = |E|$

c) Prove that any laminar family on a universe of $|V|$ elements that excludes singleton sets has size at the most $|V| - 1$.

Use all the above to finish the proof.

(You do not need to spell out all the details. As long as you can convince me you have understood the solution, you earn credit)