Lecture 3: Conditional Probability and Stochastic Independence

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Conditional Probability

- Probability given an event (B) has occurred
- Revision of belief after knowing about an event
- Definition:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{1}$$

- Equivalent to bringing a new sample space (B) whose subset is event A.

Remark

Knowing about B can actually decrease the probability of A.
Treat conditional probability with normal probability rules.

$$P(A \cap B \mid C) = P(A \mid C) + P(B \mid C), if, A \cap B = \varnothing$$

Example

- Let B be the event: $\min(X, Y) = 2$
- Let M= $\max(X, Y)$, $P(M = 1 \mid B)$ and $P(M = 2 \mid B) =$
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More Examples

- Four balls are placed successively (randomly) in four cells. Given that first two balls are in different cells, what is the probability that one cell contains exactly three balls?
- A family has two children. Given that the family has a boy, what is the probability that both the children are boys?

Total Probability Theorem

- Let the sample space be divided into H_i, $i = 1, 2, \ldots N$ exclusive events.
- Then any event A $= \bigcup_{i=1}^{n} A \cap H_i$

$$P(A) = \sum_i P(A \mid H_i)\, P(H_i) \tag{2}$$

Bayes Rule: Revising Beliefs

- Prior probabilities $P(A_i)$
  - Initial Beliefs.
- We know $P(B \mid A_i)$ for each i
- Wish to compute $P(A_i \mid B)$
  - revise beliefs, given that B occurred.

$$
\begin{aligned}
P(A_i \mid B) &= \frac{P(A_i \cap B)}{P(B)} \\
&= \frac{P(A_i)\, P(B \mid A_i)}{\sum_j P(A_j)\, P(B \mid A_j)}
\end{aligned} \tag{3}
$$

Stochastic Independence

- When $\mathbf{P}(A \mid B) = \mathbf{P}(A)$, beliefs for event A did not change when B occurred.
- If above eqn holds, events A and B are said to be independent.

$$\text{Defn}: \quad \mathbf{P}(A \cap B) = \mathbf{P}(A)\,\mathbf{P}(B) \tag{4}$$

Conditioning may affect independence

- Conditional independence, given C, is defined as usual $\mathbf{P}(. \mid C)$
- Assume A and B are independent
- If we are told that C occurred, are A and B independent?

Example: Conditioning may affect independence

- Two unfair coins, A and B
  $\mathbf{P}(H \mid A) = 0.9$, $\mathbf{P}(H \mid B) = 0.1$.
- Either coin picked with equal probability
- Once we know it is coin A, are tosses independent?
- If we do not know which coin it is, are tosses independent?
  - $\mathbf{P}(\text{toss11} = H)$
  - $\mathbf{P}(\text{toss11} = H \mid \text{first 10 tosses are heads})$

Independence of many events

- Collection of events are independent if information of "some" set of events does not give any information about some other set.
- Mathematical definition:
  Events $A_1, A_2, \ldots A_N$ are **independent if**

$$\mathbf{P}\left(A_i \cap A_j \cdots \cap A_q\right) = \mathbf{P}\left(A_i\right)\mathbf{P}\left(A_i\right)\ldots\mathbf{P}\left(A_i\right) \tag{5}$$

  For all possible sets of indices $i, j, \ldots q$

Independence vs Pairwise independence

- Two independent coin tosses
  - Event: A First toss gives head
  - Event: B Second toss gives head
  - Event: C First and second toss gives same result
  - $\mathbf{P}(C) =$
  - $\mathbf{P}(C \cap A) =$
  - $\mathbf{P}(C \cap A \cap B) =$
- Pairwise independence does not imply joint independence!