Random Processes

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Bernoulli Process

- A sequence of independent Bernoulli trials
- At each trial i
  1. $\mathbf{P}(\textit{success}) = \mathbf{P}(X_i = 1) = p$
  2. $\mathbf{P}(\textit{failure}) = \mathbf{P}(X_i = 0) = 1 - p$
- Examples:
  1. Sequence of lottery wins or losses
  2. Arrivals (each second) to a bank
  3. Arrivals (at each time slot) to server

Random processes

- Can be treated as a sequence of random variables $X_1, X_2 \ldots$
- The objective is to understand and characterise the interaction among the individual variables
- Random processes we will study:
- Bernoulli process (memoryless, discrete time)
- Poisson process (memoryless, continuous time)
- Markov chains (with memory/dependence across time)

Number of successes S in n time slots

- $\mathbf{P}(S = k)$
- $\mathbf{E}[S]$
- $Var(S)$

Interarrival Times

- T_1: number of trials until first success
- $\mathbf{P}(T_1) =$
- $\mathbf{E}[T_1]$
- $var(T_1)$
- Memoryless Property
- If you buy a lottery ticket every day, what is the distribution of the length of the first string of losing days

Time of the k^{th} arrival

- Given that first arrival was at time t, i.e $T_1 = t$, additional time, T_2, until next arrival
  1. Has the same Geometric distribution
  2. Independent of T_1
  3. Y_k: number of trials to k^{th} success
     - 3.1 $\mathbf{E}[Y_k]$
     - 3.2 $var(Y_k)$
     - 3.3 $\mathbf{P}(Y_k = t)$

Poisson Process

- **Time Homogeneity**

  $p(k, t) =$ probability of k arrivals in interval of duration t
- Number of arrivals in disjoint time intervals are independent
- Small interval probabilities:

$$p(k, \delta) \approx \lambda\delta, \quad k = 1$$

$$p(k, \delta) \approx 1 - \lambda\delta, \quad k = 0$$

- λ is called arrival rate

PMF of Number of Arrivals N

- Basic Idea: Poisson is a finely discretized Bernoulli Process
- N_t (of discrete approximation): Binomial
- Taking $\delta \to 0$, gives:

$$p(k,t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad , k = 0, 1, \ldots \tag{1}$$

- $E[N_t] = \lambda t$, $var(N_t) = \lambda t$

Example

- You get email according to a Poisson process at a rate of $\lambda = 5$ messages per hour. You check your email every thirty minutes.
- What is the probability that you received no messages?
- What is the probability that you received one new message?

Interarrival Times

- Y_k time of k^{th} arrival
- Erlang Distribution:

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$$

- Time of first arrival ($k = 1$) is exponential

$$f_{Y_1}(y) = \lambda e^{-\lambda y}$$

- Memoryless Property: Time to the next arrival is independent of the past

$$\mathbf{P}\left(Y > t + x) \mid Y > x\right) = \mathbf{P}\left(Y > t\right)$$

Used bulb is no better or worse than a new bulb

Splitting and Merging of Poisson Processes

- Recall that some of two independent Poisson r.vs with expectation λ_1 and λ_2 respectively is a Poisson r.v with expectation $\lambda_1 + \lambda_2$
- Merging and Splitting of Poisson process follow similarly as Bernoulli Process

Exercises

1. Consider a Poisson process with rate λ. Compute (a) $\mathbf{E}$ (time of the 10th event), (b) $\mathbf{P}$ (the 10th event occurs 2 or more time units after the 9th event), (c) $\mathbf{P}$ (the 10th event occurs later than time 20)
2. Customers arrive at a store at a rate of 10 per hour. Each is either male or female with probability half each. Assume that you know that exactly 10 women entered within some hour (say, 10 to 11am). Compute the probability that exactly 10 men also entered.
3. Assume that cars arrive at a rate of 10 per hour. Assume that each car will pick up a hitchhiker with probability $\frac{1}{2}$. You are second in line. What is the probability that you will have to wait for more than 2 hours.

Exercises

1. You have three friends, A, B, and C. Each will call you after an Exponential amount of time with expectation 30 minutes, 1 hour, and 2.5 hours, respectively. You will go out with the first friend that calls. What is the probability that you go out with A?
2. Each light bulb has independent exponential lifetime. Install three light bulbs. Find expected time until last light bulb dies out.