Continuous Random Variables: Bayes Rule, Functions of Random Variable(s)

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Continuous R.V: Probability Density Function (PDF)

- $F_{X,Y}(x,y) = \mathbf{P}(X \leq x, Y \leq y)$
- $f_{X,Y}(x,y) = \frac{\partial F_{X,Y}(x,y)}{\partial x \partial y}$
- Marginals from joint : $f_X(x) = \int_y f_{X,Y}(x,y)\, dy$
  Proof: start from the definition of Cumulative distribution.
- Conditional density $f_{Y \mid X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$

Application of Conditional Density for Bayesian Inference

- Bayes rule for continuous case:

$$f_{Y \mid X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{f_{X \mid Y}(x \mid y) f_Y(y)}{f_X(x)}$$

- We dont know X, but we observe Y.
- X: some signal, "prior" $f_X(x)$
- Y: noisy version of X.
- $f_{Y \mid X}(y \mid x)$: model of the noise.

Discrete X, Continuous Y

- X: a discrete signal, "prior" $p_X(x)$
- Y: noisy version of X.
- $f_{Y \mid X}(y \mid x)$: model of the noise.

$$p_{X \mid Y}(x \mid y) = \frac{f_{Y \mid x}(y \mid x) P_X(x)}{f_Y(y)}$$

$$f_Y(y) = \sum_x p_X(x) f_{Y \mid X}(y \mid x) \tag{1}$$

Discrete Y, Continuous X

- X: a continuous signal, "prior" $f_X(x)$
  e.g: temperature on a given day
- Y: noisy version of X.
  e.g: rainfall on that day.
- $P_{Y \mid X}(y \mid x)$: model of the noise.

$$f_{X \mid Y}(x \mid y) = \frac{p_{Y \mid x}(y \mid x) f_X(x)}{p_Y(y)}$$

$$p_Y(y) = \int_x f_X(x) p_{Y \mid X}(y \mid x)\, dx \tag{2}$$

Functions of Random Variables: Derived Distributions

- It is a PMF or PDF of a function of one or more random variables with known probability law.
- E.g: Obtaining the pdf of $Z = \frac{X}{Y}$, if X, Y are jointly uniform (and independent).
- Two cases: Discrete and Continuous

The Continuous Case

- Two step procedure
  - Get C.D.F of Y: $F_Y(y) = \mathbf{P}\left(Y \leq y\right)$
  - Differentiate that to get

$$f_Y(y) = \frac{\partial F_Y}{\partial y}$$

- Example 1: X is uniform in $[0, 2]$ Find the pdf of $Y = X^3$
  - First find $\mathbf{P}\left(X^3 < y\right)$..
- Example 2: If your driving speed is uniform between 30 and 60 kmph. What is the P.D.F of the time taken to travel 200 kms ?
- The pdf of $aX + b$ is $\frac{1}{|a|} f_X(\frac{y-b}{a})$

Function of two r.vs

- Find the P.D.F of $Z = \frac{Y}{X}$
- If X, Y are jointly uniform and independent in a unit square.
  1. $F_Z(z)$ when $z < 1$
  2. $F_Z(z)$ when $z > 1$
- What is the P.D.F of $Z = X + Y$ for any independent X and Y
- $f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z - x)\, dx$

Sum of two Normal Random Variables

- Consider $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$
- If X and Y are independent
- Then $Z = X + Y$ is also normal with mean$= \mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$

Sum of Random number of independent r.v.'s

- N: Number of stores visited
- X_i money spent in store i
- X_i independent of N.
- Let $Y = X_1 + X_2 \dots X_N$

$$\mathbf{E}[Y \mid N = n] = n\mathbf{E}[X]$$

- What is $\mathbf{E}[Y]$

Characteristic Functions

- Like a Fourier Transform of the P.D.F.
- Define the characteristic function of a random variable X to be the complex-valued function on $t \in R$ as

$$\psi_X(t) = \mathbf{E}[e^{itX}] = \int e^{itx} f_X(x)\, dx \tag{3}$$

Properties of the Characteristic Function

- ψ exists for any distribution on X
- $\psi(0) = 1$
- $|\psi(t)| \leq 1$
- The characteristic function of $a + bX$ is
- The characteristic function of the convolution of two r.vs' is