Continuous Random Variables: PDF,CDF

IIIT Delhi

praveshb@iiitd.ac.in

September 27, 2016

Continuous R.V: Probability Density Function (PDF)

- A continuous R.V is described by a **PDF** f_X

$$\mathbf{P}\left(a \leq x \leq b\right) = \int_a^b f_X(x)\,dx$$

$$\int_{-\infty}^{\infty} f_X(x) = 1$$

$$\mathbf{P}\left(x \leq x \leq x + \delta\, x\right) \approx f_X(x)\delta\, x$$

$$\mathbf{P}\left(x \in B\right) = \int_B f_X(x)dx$$

- Set B should be a union of "measurable" sets or union of intervals.
- Gives the notion of "density".

Mean and Variances

- $\mathbf{E}[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$
- $\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$
- $var(X) = \int_{-\infty}^{\infty} (x - \mathbf{E}[X])^2\, f_X(x)\, dx$
- Example: Continuous Uniform R.V: $f_X(x) = \frac{1}{b-a}$, when $a \leq x \leq b$, otherwise $f_X(x) = 0$
- What is the mean and Variance?

Cumulative Distribution Function (CDF)

- Unifies the continuous and discrete random variable.
- Define $F_X(x) = \mathbf{P}\,(X \leq x)$
- Monotonically non-decreasing function of X.
- Draw the C.D.F of Uniform r.v
- $F(x + \delta\, x) - F(x) \approx f\,(x)\,\delta x$

Joint PDF $f_{X,Y}(x, y)$

- Total Volume $= 1$, given as $\int_x \int_y f_{X,Y}(x, y)\, dx\, dy = 1$
- Expectation: $\mathbf{E}[g(X, Y)] = \int_x \int_y g(x, y)\, f_{X,Y}(x, y)\, dx\, dy = 1$
- From the joint to the marginal:
  $\mathbf{P}\left(x \leq X \leq x + \delta\right) = \int_{x=x}^{x=\delta+x} \int_{y=-\infty}^{y=\infty} f_{X,Y}(x, y) = f_X(x)\, \delta$
- X and Y are independent if $f_{X,Y}(x, y) = f_X(x) f_Y(y)$

Example: Buffon's Needle

- Consider many parallel lines with distance d
- A needle of length $l < d$ is thrown on this lines
- Find Probablity that the Needle does not intersect any line
- Steps:
  - Find the r.vs.: $0 < x < \frac{d}{2}$, $0 < \theta < \frac{\pi}{2}$
  - Find the probabilty model: $f_{X,\Theta}(x, \theta)$ is uniform.
  - Find the Event space:
  - Calculate the probability

Conditioning

- Recall: $\mathbf{P}\left(x \leq X \leq x + \delta\right) = f_x(X)\,\delta$
- Similarly: $\mathbf{P}\left(x \leq X \leq x + \delta \mid Y \approx y\right) \approx f_{X \mid Y}(x|y)\,\delta$
- Definition:

$$f_{X \mid Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

- Same as normalising the joint PDF by the density at a given y.
- Independence: