Lecture 2: Counting

IIIT Delhi

praveshb@iiitd.ac.in

August 10, 2016

Basic Principles of Counting

- r stages
- n_i choices at stage i
- Total number of choices is: $n_1 n_2 \ldots n_r$
- Example: How many license plates with 3 letters and 4 digits?
- Example: How many ways r balls can be placed in n cells?
- Example: How many subsets from a set $S = \{1, 2, \ldots, n\}$
- What if no repetition is allowed(first example)?

Ordered Samples: Permutations

- For a population of n elements, and a prescribed sample size r, there exists n^r samples with replacement and $(n)_r$ without replacement.
- Number of ways of ordering n elements is $n!$

Ordering: Examples with Probability Calculation

- In sampling without replacement what is the probability for any fixed element of the population to be included in a sample of size r?
- If n balls are randomly placed in n cells, the probability that each cell will be occupied is?
- In a room filled with r people, what is the probability that no group has a common birthday?

Partitions: Combinations

- $\binom{n}{k}$: number of k element subsets of a given n element set.
- When order is not important!
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
  1. Pick any k elements from the set and make $k!$ arrangements.
  2. $(n)_k$ total arrangements.
  3. $\binom{n}{k} = \frac{(n)_k}{k!}$
- Also called the binomial coefficient.

Partitions: Examples

- How many ways can we pick r elements from a n element set?
- *Occupancy Problem*: Consider a random allotment of r balls in n cells. What is the Prob. that a specified cell contains exactly k balls?
- Consider a set with p indistinguishable elements of one type and q indistinguishable elements of another type. What is the number of ways in which the set can be arranged?

Partitions: Occupancy Problems

Consider r indistinguishable balls to be put in n cells such that r_i is the number of balls in the i^{th} cell.

$$r_1 + r_2 + \ldots r_n = r \tag{1}$$

- The number of distinguishable distributions (that is the number of different solutions of (1)
  is
  $$A_{r,n} = \binom{n+r-1}{r}$$
- The number of distinguishable distributions in which no cell remains empty is $\binom{r-1}{n-1}$

Indistinguishable Objects: More Examples

- There are $\binom{r+5}{5}$ distinguishable results of a throw with r indistinguishable dice.
- `Bose Einstein and Fermi Dirac statistics`:
  The probability that cells numbers $1, 2, \ldots n$ contain $r_1, r_2 \ldots r_n$ balls, respectively (where $\sum r_i = r$) is given by $\frac{1}{A_{r,n}}$ and it is equal to $\binom{n}{r}^{-1}$ under Fermi-Dirac statistics provided r_j equals 0 or 1.

Stirling's Formula

$$n! \approx \sqrt{2}\, n^{n+\frac{1}{2}}\, e^{-n}$$

- 1! is 0.9221
- The percentage error decreases rapidly as the value of n increases.