Markov Chains

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Checkout Counter Model

- Discrete time $n = 0, 1 \ldots$
- Customer Arrivals: Bernoulli(p), geoemtric interarrival
- Customer Service times: Geometric(q)
- State X_n: number of customers at time n.

Finite State Markov Chains

- X_n: state after n transitions
- Each state belongs to finite set, $1, 2 \dots m$
- Markov property: given current state, the past does not matter

$$p_{ij} = \mathbf{P}\left(X_{n+1} = j \mid X_n = i\right)$$

$$p_{ij} = \mathbf{P}\left(X_{n+1} = j \mid X_n = i, X_{n-1}, \dots X_0\right)$$

- Model specifications
  - Identify the possible states
  - Identify the possible transitions
  - Identify the transitions probabilities

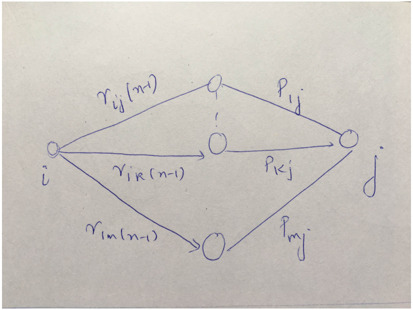
n-step transition probabilities

- State occupancy probabilities, given initial state i

$$r_{ij}(n) = \mathbf{P}\left(X_n = j \mid X_0 = i\right)$$

- Key recursion: $r_{ij}(n) = \sum r_{ik}(n-1)p_{kj}$
- With random initial state

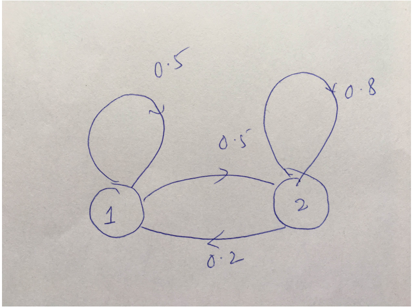
$$\mathbf{P}\left(X_n = j\right) = \sum_{i=1}^{m} \mathbf{P}\left(X_0 = i\right) r_{ij}(n)$$



An Example

A wireless packet communications channel suffers from clustered errors. That is, whenever a packet has an error, the next packet will have an error with probability 0.9. Whenever a packet is error free, the next packet is error free with probability 0.99. When $X_n = 1$ if the n^{th} packet has an error; otherwise, $X_n = 0$. Sketch the chain and find the transition probability matrix.

Example 2



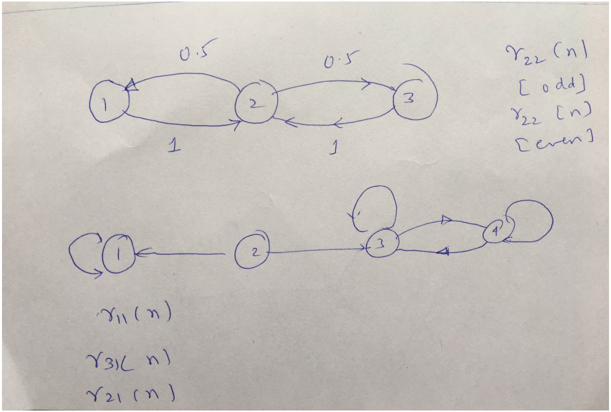
Chapman-Kolmogorov Equations

For a finite Markov chain with transition matrix $\mathbf{P}$, the n step transition matrix is

$$\mathbf{P}(n) = \mathbf{P}^n \tag{1}$$

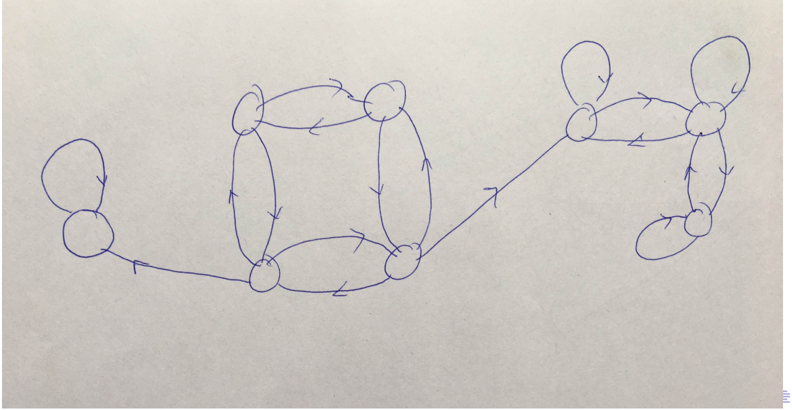
Generic Convergence Questions

- Do the $r_{ij}(n)$ converge to π_j (independent of i).
- Does the limit depends on the initial state?



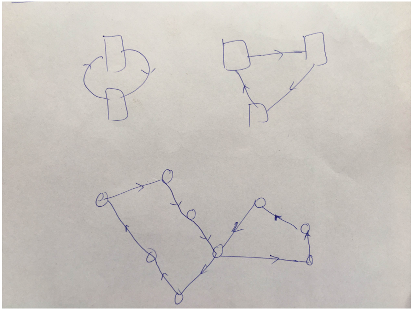
Recurrent and Transient States

- State i is recurrent, if: starting from i, and from wherever you can go, there is a way of returning to i
- If not recurrent, called transient.
- Recurrent Class: Collection of recurrent states that communicate with each and with no other state.



Periodic States

- The states in a recurrent class are periodic if they can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group.
- Self transition is not periodic.



Steady State Probabilities

- Do the $r_{ij}(n)$ converge to π_j?
- Yes, if:
  - recurrent states are all in a single class, and,
  - Single recurrent class is not periodic
- Assuming, yes, start with from key recursion

$$r_{ij}(n) = \sum r_{ik}(n-1)p_{kj}$$

- Take limit $n \to \infty$

$$\pi_j = \sum \pi_k p_{kj}$$

- additional equation $\sum \pi_j = 1$

Birth Death Process