PRP Assignment 4

1. Let $X$and Y be independent random variables with common distribution function F and density function f. Show that $V = \max\{X, Y\}$ has distribution function $\mathrm{P}(V \leq x) = F(x)^2$ and density function $f_v(x) = 2\, f(x)\, F(x)$, x $\in R$. Find the density function of $U = \min\{X, Y\}$.
   If X and Y is uniformly distributed between 0 and 1. Let $Z = \max\{X, Y\}$ determine the CDF and mean of Z.

2. Let X be a random variable with mean μ and continuous distribution function F. Show that

$$\int_{-\infty}^{a} F(x)\, dx = \int_{a}^{-\infty} [1 - F(x)]\, dx$$

   if and only if a=μ.

3. Let A be the circle $\{(a, b)$ such that $a^2 + b^2 \leq 1\}$. A point p is chosen randomly on the boundary of the circle and another point q is chosen randomly from the interior of circle (these points are chosen independently and uniformly over their domains). Let B be the rectangle with sides parallel to the x-axis and y-axis with diagonal $p\,q$. What is the probability that no point of B lies outside of A (circle)?

4. Let X be uniformly distributed on $[0\ \frac{\pi}{2}]$. Find the density function of $Y = \sin X$.

5. Find the conditional density function and expectation of Y given X when they have joint density function:
   - $f(x, y) = \lambda^2 e^{-\lambda y}$ for $0 \leq x \leq y < \infty$.
   - $f(x, y) = x e^{-x(y+1)}$ for $x, y \geq 0$.

6. Prove that $\mathrm{Var}(X) = \mathrm{E}[\,\mathrm{Var}(X|Y)\,] + \mathrm{Var}[\,\mathrm{E}(X|Y)\,]$.