1. Prove that a list of vectors is linearly dependent if and only if at least one of the vectors is a linear combination of the others.

Sol^n: Suppose $\bar{v}_1, \dots \bar{v}_p$ are linearly dependent vectors. Then, there exist scalars $c_1, c_2 \dots c_p$, not all zero, such that

$$c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_p\bar{v}_p = 0 \quad \text{——— (1)}$$

Without loss of generality, take $c_1 \neq 0$

$$\therefore c_1\bar{v}_1 = -c_2\bar{v}_2 + \dots + c_p\bar{v}_p$$

$$\bar{v}_1 = (-c_1^{-1}c_2)\bar{v}_2 + (-c_1^{-1}c_3)\bar{v}_3 + \dots + (-c_1^{-1}c_p)\bar{v}_p$$

ie, $\bar{v}_1$ is a linear combination of the other vectors.

CONVERSE:

Suppose, $\bar{v}_1, \bar{v}_2 \dots \bar{v}_p$ is a list of vectors where at least one vector is a linear combination of the others.

Let
$$\bar{v}_p = c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_{p-1}\bar{v}_{p-1} \quad \text{——— (2)}$$

for some scalars $c_1, c_2 \dots c_{p-1}$.

(2) $\Rightarrow$ $c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_{p-1}\bar{v}_{p-1} + (-1)\bar{v}_p = 0$

In the above expression, coefficient of $\bar{v}_p$ is $-1 \neq 0$.

$\therefore$ By definition, the list is linearly dependent.

2. Show that if U and W are subspaces of the vector space V, then $U+W = \{\bar{u}+\bar{w} : \bar{u} \in U, \bar{w} \in W\}$ is also a subspace of V. Furthermore, show that $U+W$ is the smallest subspace of V that contains both U and W.

Solⁿ: Test for subspaces:

i) zero vector must lie in the subspace –

$\bar{0} \in \bar{u}$, $\bar{0} \in \bar{w}$

$\Rightarrow \bar{0} = \bar{0} + \bar{0} \in U+W$

ie, $\bar{0}$ vector lies in $U+W$.

ii) Closure under addition –

Suppose $\bar{v}_1, \bar{v}_2 \in U+W$

Then, $\bar{v}_1 = \bar{u}_1 + \bar{w}_1$

$\bar{v}_2 = \bar{u}_2 + \bar{w}_2$, where $\bar{u}_1, \bar{u}_2 \in U$, $\bar{w}_1, \bar{w}_2 \in W$

$\therefore \bar{v}_1 + \bar{v}_2 = (\bar{u}_1 + \bar{w}_1) + (\bar{u}_2 + \bar{w}_2)$

$= (\bar{u}_1 + \bar{u}_2) + (\bar{w}_1 + \bar{w}_2)$

$= \bar{u}_3 + \bar{w}_3$, where $\bar{u}_3 \in U$, $\bar{w}_3 \in W$

$\therefore \bar{v}_1 + \bar{v}_2 \in U+W$

iii) Closure under scalar multiplication –

Let c be a scalar

$c\bar{v}_1 = c(\bar{u}_1 + \bar{w}_1)$

$= c\bar{u}_1 + c\bar{w}_1$

$= \bar{u}_4 + \bar{w}_4$, $\bar{u}_4 = c\bar{u}_1 \in U$, $\bar{w}_4 = c\bar{w}_1 \in W$

$\therefore c\bar{v}_1 \in U+W$

Hence, by the three properties, $U+W$ is a subspace of V.

Let, X be a subspace of V that contains both U and W

ie, $U \subseteq X$ and $W \subseteq X$.

Let $\bar{v} = \bar{u} + \bar{w}$ be any element of $U+W$

Then, $\bar{u} \in X$ and $\bar{w} \in X$

$\Rightarrow \bar{u} + \bar{w} \in X$

$\therefore U+W \subseteq X$

Since, $U+W$ is contained in X, hence $U+W$ is the smallest subspace containing both U and W.

3. Given any two $m \times n$ matrices A and B, prove that

$$\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B).$$

Soln

$$\text{rank}(A+B) = \dim\{\text{col}(A+B)\}$$

$$\cancel{= \dim\{\text{col}(A)}$$

Since, $(A+B)x = Ax + Bx$

$\therefore \text{col}(A+B) \subseteq \text{col}(A) + \text{col}(B)$

$$\begin{aligned}\therefore \text{rank}(A+B) &= \dim\{\text{col}(A+B)\} \\ &= \dim\{\text{col}(A)\} + \cancel{\text{col}(B)} \dim\{\text{col}(B)\} - \dim\{\text{col}(A \cap B)\} \\ &\leq \dim\{\text{col}(A)\} + \dim\{\text{col}(B)\} \\ &= \text{rank}(A) + \text{rank}(B)\end{aligned}$$

$\Rightarrow \text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$.

4. Let λ be an eigenvalue of an invertible matrix A, show that λ^{-1} is an eigenvalue of A^{-1}.

Sol^n Since λ is an eigenvalue of A,

$$\exists\ \bar{x} \text{ s.t. } \quad A\bar{x} = \lambda\bar{x} \qquad (\bar{x} \neq \bar{0})$$

Since A is invertible,

$$A^{-1}A\bar{x} = A'\lambda\bar{x}$$

$$\Rightarrow \bar{x} = \lambda(A^{-1}\bar{x})$$

Now, $\lambda \neq 0$ (since A is invertible)

$$\therefore \quad \lambda^{-1}\bar{x} = A^{-1}\bar{x}$$

Hence, λ^{-1} is an eigenvalue of A'.

5. $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$

(a) Find the SVD of A.

$A^TA = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

eigenvalues of $A^TA \rightarrow |A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda = 2, 6$

eigenvectors:

$(A - \lambda_1 I)\bar{x}_1 = 0$

$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \bar{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

orthonormal $\rightarrow \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$(A - \lambda_2 I)\bar{x}_2 = 0$

$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = 0$

$\therefore \bar{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

orthonormal $\rightarrow \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\therefore V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ (in descending order of eigenvalues)

Similarly, $AA^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 5 \end{bmatrix}$

eigenvalues of AA^T are $\rightarrow$ 0, 0, 2, 6 eigenvector corresponding to $\lambda = 2$:

$$(A-\lambda I)\bar{x} = \bar{0}$$

$$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad \bar{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \xrightarrow{\text{orthonormal}} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

eigenvector corresponding to $\lambda = 6$:

$$(A-\lambda I)x = \bar{0}$$

$$\begin{bmatrix} -5 & 1 & 1 & 1 \\ 1 & -5 & 1 & 1 \\ 1 & 1 & -5 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \bar{x} = \bar{0}$$

$$\therefore \quad \bar{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} \xrightarrow{\text{orthonormalize}} \begin{bmatrix} 1/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \\ 3/\sqrt{12} \end{bmatrix}$$

eigenvectors corresponding to $\lambda = 0$

$$(A-\lambda I)\bar{x} = \bar{0}$$

$$A - \lambda I = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \therefore \quad \begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

$$\text{Hence,} \quad \bar{x} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix} \xrightarrow{\text{orthonormalize}} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \end{bmatrix}$$

$$\therefore\ U = \begin{bmatrix} \frac{1}{\sqrt{12}} & 1/2 & 0 & -2/\sqrt{6} \\ 1/\sqrt{12} & 1/2 & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{12} & 1/2 & -1/\sqrt{2} & 1/\sqrt{6} \\ 3/\sqrt{12} & -1/2 & 0 & 0 \end{bmatrix}$$

Also, $\Sigma = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ (non zero eigenvalues of AA^T and AA^T arranged in descending order)

$$\therefore\ A = \underbrace{\begin{bmatrix} 1/\sqrt{12} & 1/2 & 0 & -2/\sqrt{6} \\ 1/\sqrt{12} & 1/2 & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{12} & 1/2 & -1/\sqrt{2} & 1/\sqrt{6} \\ 3/\sqrt{12} & -1/2 & 0 & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{V^T}$$

(b). Find a basis for Col A and Row A.

Sol^n rank(A) $= r = 2$

Basis for Col A $= \{\bar{u}_1, \dots \bar{u}_r\} = \{\bar{u}_1, \bar{u}_2\}$

$$= \left\{ \begin{bmatrix} 1/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \\ 3/\sqrt{12} \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \right\}$$

Basis for Row A $= \{\bar{v}_1, \dots \bar{v}_r\} = \{\bar{v}_1, \bar{v}_2\}$

$$= \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}$$

(c) Find a basis for Nul A and Nul A^T.

Basis for Nul A $= \{ \bar{v}_{r+1}, \dots \bar{v}_n \}$ $\quad [\underset{m\times n}{A} = \underset{m\times m}{U} \underset{m\times n}{\Sigma} \underset{n\times n}{V^T}]$

$= \phi$

$\therefore$ Nul space of A does not exist

Basis for Nul A^T $= \{ \bar{u}_{r+1}, \dots \bar{u}_m \}$

$= \{ \bar{u}_3, \bar{u}_4 \}$

$$= \left\{ \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \end{bmatrix} \right\}$$

6/ Find the Hessian of the function

$$f(x_1, x_2, x_3) = x_1^4 + (x_1+x_2)^2 + (x_1+x_3)^2$$

Is the Hessian PSD?

Solⁿ:-

$$\frac{\nabla f}{\nabla \bar{x}} = \begin{bmatrix} 4x_1^3 + 2(x_1+x_2) + 2(x_1+x_3) \\ 2(x_1+x_2) \\ 2(x_1+x_3) \end{bmatrix}$$

$$H = \frac{\nabla^2 f}{\nabla x^2} = \begin{bmatrix} 12x_1^2+4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

H will be PSD if and only if the determinants of all the leading principal minors are non-negative.

$$|H_1| = 12x_1^2 + 4 > 0$$

$$|H_2| = \begin{vmatrix} 12x_1^2 & 2 \\ 2 & 2 \end{vmatrix} = 12x_1^2 \geq 0$$

$$|H_3| = \begin{vmatrix} 12x_1^2+4 & 2 & 2 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = (12x_1^2+4)4 - 2\times 4 - 2\times 4$$

$$= 48x_1^2 \geq 0$$

Hence, the Hessian is PSD.

7. Show from the eigenvalues that if A is positive definite, so are A^2 and A^{-1}.

Sol^n If A has eigenvalue λ with corresponding eigenvector $\bar{x}$, then A^{-1} has eigenvalue λ^{-1} corresponding to the same eigenvector $\bar{x}$.

Since A is ~~Pos~~ PD

$$\lambda > 0 \qquad (\text{for all eigenvalues of } A)$$

$$\Rightarrow 1/\lambda \cancel{\geq 0} > 0$$

$\Rightarrow$ all eigenvalues of A^{-1} are $\cancel{\geq 0}$ > 0

$\Rightarrow$ A^{-1} is PD.

If A has eigenvalue λ corresponding to eigenvector $\bar{x}$, A^2 has eigenvalue λ^2 corresponding to same eigenvector $\bar{x}$.

Since A is PD,

$$\lambda > 0$$

$$\Rightarrow \lambda^2 > 0$$

$\Rightarrow$ all eigenvalues of A^2 are > 0

$\Rightarrow$ A^2 is PD.

8| Prove that if A is an $n \times n$ square matrix, then

$|A| = \pm \sigma_1 \sigma_2 \dots \sigma_n$.

Soln: Let, $A = U \Sigma V^T$ be a singular value decomposition of A.

Then, $|A| = |U| \, |\Sigma| . |V^T|$

Since U and V are orthogonal matrices,

$|U| = \pm 1$ and $|V^T| = \pm 1$

$\therefore \; |A| = \pm |\Sigma|$

Since Σ is square, it must be diagonal matrix with diagonal elements $\sigma_1, \sigma_2, \dots, \sigma_n$ (singular values of A).

$\therefore \; |\Sigma| = \sigma_1 \sigma_2 \dots \sigma_n$

$\Rightarrow |A| = \pm \sigma_1 \sigma_2 \dots \sigma_n$.

9/. For what values of k, will the following matrices be PD?

(a) $A = \begin{bmatrix} 2 & -4 \\ -4 & k \end{bmatrix}$

Soln: A 2x2 real symmetric matrix is positive definite if and only if its diagonal entries are positive and if its determinant is positive

$\therefore \; k > 0$ — (1)

and $|A| = 2k - 16 > 0$

$\Rightarrow k > 8$

Hence, $\boxed{k > 8}$.

(b) $A = \begin{bmatrix} k & 5 \\ 5 & -2 \end{bmatrix}$

Soln: A can never be positive definite, since it has a negative diagonal entry -2.

10/. Suppose that A and B are positive definite matrices. Show that

(a). $A+B$ is positive definite.

Soln: Since A and B are positive definite,

$$\bar{x}^T A \bar{x} > 0 \quad \text{——} ①$$

$$\bar{x}^T B \bar{x} > 0 \quad \text{——} ②$$

$① + ② \Rightarrow \quad \bar{x}^T (A+B) \bar{x} > 0$

Hence $A+B$ is positive definite.

(b). KA is positive definite, for $K>0$.

Soln:

$$x^T A x > 0$$

$$\therefore Kx^T A x > 0 \qquad (K>0)$$

$$\Rightarrow x^T (KA) x > 0$$

Hence, KA is positive definite.

11/ Let $A \in M_n$ be self-adjoint. Show that
$$U = (I - iA)(I + iA)^{-1} \text{ is unitary.}$$

Sol^n:

$$U = (I - iA)(I + iA)^{-1}$$
$$U^H = ((I + iA)^{-1})^H (I - iA)^H$$
$$= (I - iA^H)^{-1}(I + iA^H)$$
$$= (I - iA)^{-1}(I + iA) \qquad [A = A^H \text{ (self-adjoint)}]$$

$$\therefore U^H U = (I - iA)^{-1}(I + iA)(I - iA)(I + iA)^{-1} \text{ ——— (1)}$$

Now, $(I + iA)(I - iA) = I + A^2$
$(I - iA)(I + iA) = I + A^2$ $\Big\} \Rightarrow (I + iA)(I - iA)$ can be commuted

$$\therefore (1) \Rightarrow U^H U = (I - iA)^{-1}(I - iA)(I + iA)(I + iA)^{-1}$$
$$= I$$

Hence, U is unitary.

12/ Let $m \leq n$, $A \in M_n$, $B \in M_m$, $Y \in M_{n \times m}$ and $Z \in M_{m \times n}$. Assume that A and B are invertible. Show that $A + YBZ$ is invertible if and only if $B^{-1} + ZA^{-1}Y$ is invertible. Moreover,
$$(A + YBZ)^{-1} = A^{-1} - A^{-1}Y(B^{-1} + ZA^{-1}Y)^{-1}ZA^{-1}$$

Sol^n:
$$P + PQP = P(I + QP) = (I + PQ)P$$
$$\Rightarrow (I + PQ)^{-1}P = P(I + QP)^{-1}$$
$$\Rightarrow (I + PQ)^{-1} = P(I + QP)^{-1}P^{-1}$$
$$\cancel{= P[I - (I -}$$
$$= P(I + QP)^{-1}(I + QP - QP)P^{-1}$$
$$= P[I - (I + QP)^{-1}QP]P^{-1}$$
$$= [P - P(I + QP)^{-1}QP]P^{-1}$$
$$\therefore (I + PQ)^{-1} = I - P(I + QP)^{-1}Q \text{ ——— (1)}$$

Now, $(A+YBZ)^{-1} = ((I+YBZA^{-1})A)^{-1}$

$$= A^{-1}(I+YBZA^{-1})^{-1}$$

$$= A^{-1}[I - YB(I+ZA^{-1}YB)^{-1}ZA^{-1}] \qquad (\text{from } ①)$$

$$= A^{-1} - A^{-1}YB(I+ZA^{-1}YB)^{-1}ZA^{-1}$$

$$= A^{-1} - A^{-1}Y[(I+ZA^{-1}YB)B^{-1}]^{-1}ZA^{-1}$$

$$\therefore (A+YBZ)^{-1} = A^{-1} - A^{-1}Y(B^{-1}+ZA^{-1}Y)^{-1}ZA^{-1}$$

Since, the computation of $(A+YBZ)^{-1}$ involves the computation of $(B^{-1}+ZA^{-1}Y)^{-1}$, hence $(A+YBZ)$ is invertible if and only if $(B^{-1}+ZA^{-1}Y)$ is invertible.

13/. The $n \times n$ Pascal matrix is defined as

$$P_{ij} = \binom{i+j-2}{i-1} \qquad (1 \le i, j \le n)$$

What is the determinant?

<u>Sol^n</u>: The 3×3 Pascal matrix is

$$A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

The LU decomposition of A yields

$$A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly for the 4x4 Pascal matrix,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Generalizing the above matrices,

$$P = L\,U$$

where, $P_{ij} = {}^{i+j-2}C_{i-1}$

$L_{ij} = {}^{i-1}C_{j-1}$

$U_{ij} = {}^{j-1}C_{i-1}$

Hence, each element P_{ij} of P can be written as —

$$P_{ij} = {}^{i+j-2}C_{i-1} = \sum_{k=0}^{n-1} \left\{ {}^{i-1}C_k \times {}^{j-1}C_k \right\}$$

which is a proven theorem.

Hence, every Pascal matrix can be decomposed into a lower triangular and an upper triangular matrix with diagonal elements = 1.

$$\therefore \quad |P_{ij}| = |L| \times |U|$$

$$= \left\{ \prod_{i=1}^{n} {}^{i-1}C_{i-1} \right\} \times \left\{ \prod_{i=1}^{n} {}^{i-1}C_{i-1} \right\}$$

$$= 1 \times 1$$

$$= 1.$$