Engineering Optimization

ECE305/ECE505/MTH3EO/MTH5EO

Assignment 1

August 20, 2015

1. Prove that a list of vectors is linearly dependent if and only if at least one of the vectors is a linear combination of the others.

2. Show that if U and W are subspaces of the vector space V, then $U+W = \{\mathbf{u}+\mathbf{w} : \mathbf{u} \in U, \mathbf{w} \in W\}$ is also a subspace of V. Furthermore, show that $U+W$ is the smallest subspace of V that contains both U and W.

3. Given any two $m \times n$ matrices A and B, prove that $rank(A+B) \leq rank(A) + rank(B)$.

4. Let λ be an eigenvalue of an invertible matrix A, show that λ^{-1} is an eigenvalue of A^{-1}.

5. Let
$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$$
   (a) Find the SVD of A.
   (b) Find a basis for $ColA$ and $RowA$.
   (c) Find a basis for $NulA$ and $NulA^T$.

6. Find the Hessian of the function $f(x_1, x_2, x_3) = x_1^4 + (x_1+x_2)^2 + (x_1+x_3)^2$. Is the Hessian PSD?

7. Show from the eigenvalues that if A is positive definite, so are A^2 and A^{-1}.

8. Prove that if A is an $n \times n$ square matrix, then $|A| = \pm\sigma_1\sigma_2...\sigma_n$.

9. For what values of k ,will the following matrices be positive definite :

(a)

$$A = \begin{bmatrix} 2 & -4 \\ -4 & k \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} k & 5 \\ 5 & -2 \end{bmatrix}$$

10. Suppose that A and B are positive definite matrices. Show that

    (a) $A + B$ is positive definite

    (b) kA is positive definite, for $k > 0$

11. Let $A \in \mathbb{M}_n$ be self-adjoint. Show that

$$U = (I - iA)(I + iA)^{-1}$$

    is a unitary. (U is the Cayley transform of A.)

12. Let $m \le n$, $A \in \mathbb{M}_n$, $B \in \mathbb{M}_m$, $Y \in \mathbb{M}_{n\times m}$ and $Z \in \mathbb{M}_{m\times n}$. Assume that A and B are invertible. Show that $A + YBZ$ is invertible if and only if $B^{-1} + ZA^{-1}Y$ is invertible. Moreover,

$$(A + YBZ)^{-1} = A^{-1} - A^{-1}Y(B^{-1} + ZA^{-1}Y)^{-1}ZA^{-1}.$$

13. The $n \times n$ Pascal matrix is defined as

$$P_{ij} = \begin{pmatrix} i+j-2 \\ i-1 \end{pmatrix} \qquad (1 \le i, j \le n).$$

    What is the determinant ?

14. Frobenius' inequality: Let $A : \mathcal{H}_2 \to \mathcal{H}_1$, $B : \mathcal{H}_3 \to \mathcal{H}_2$ and $C : \mathcal{H}_4 \to \mathcal{H}_3$ be linear mappings. Show that

$$rank AB + rank BC \le rank B + rank ABC.$$