Worksheet 3
IQC 2025

January 31, 2025

1. A projector is defined for any orthornormal basis $\{\left|i\right\rangle, 1 \leq i \leq n\}$ as

$$P = \sum_{i=1}^{k} \left|i\right\rangle \left\langle i\right|$$

. Show that $P^2 = P$ for a projector. Also show that all the eigenvalues of a projector are either 0 or 1.

2. Which of the following is/are valid projectors?
   (a) $P = \frac{1}{2}\left|+\right\rangle\left\langle+\right| - \frac{1}{2}\left|-\right\rangle\left\langle-\right|$
   (b) $P = \frac{1}{2}\Big(\left|0\right\rangle\left\langle 0\right| + \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|\Big)$
   (c) $P = \frac{1}{\sqrt{2}}\Big(\left|0\right\rangle\left\langle 0\right| + i\left|0\right\rangle\left\langle 1\right|\Big)$

3. Prove that for any orthonormal basis $\{\left|\psi_i\right\rangle : 1 \leq i \leq n\}$ the projectors $\{\left|\psi_i\right\rangle\left\langle\psi_i\right|, 1 \leq i \leq n\}$ satisfy the completeness relation.

4. Compute the eigendecompositions of the following unitaries
   (a) Y
   (b) S
   (c) $X \otimes Z$

5. Compute the expected values of the X, Y and Z observables using the measurement formulation and via the density matrix formulation for the following states
   (a) $\left|+\right\rangle - \left|-\right\rangle$
   (b) $\alpha\left|0\right\rangle + \beta\left|1\right\rangle, \quad |\alpha|^2 + |\beta|^2 = 1$

6. What is the density matrix obtained if you take the partial trace over the second qubit of the following state

$$\left|\psi\right\rangle = \frac{1}{\sqrt{3}}(\left|00\right\rangle + \left|01\right\rangle + \left|10\right\rangle)$$

7. Give two examples of mixed states $\rho_1 = \sum_i p_i \left|\psi_i\right\rangle\left\langle\psi_i\right|$ and $\rho_2 = \sum_j q_j \left|\phi_j\right\rangle\left\langle\phi_j\right|$ such that $\left|\psi_i\right\rangle \neq \left|\phi_j\right\rangle$ for any i and j but $\rho_1 = \rho_2$.

8. Let ρ be a density operator of a two-qubit system AB. Show that $Tr_B\big((U \otimes Y)\rho(U^\dagger \otimes Y^\dagger)\big) = UTr_B(\rho)U^\dagger$.

9. Prove that for any Hermitian operator H, the operator defined as e^{iHt} is a Unitary, where t is some arbitrary scalar.

10. Prove that the following set is a closed under matrix multiplication

$$\{\pm I, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$$

. In fact, it is a group called the Pauli Group.

11. Prove that if A is such that $A^2 = I$, then

$$e^{iAt} = \cos(t)I + i\sin(t)A$$

12. Write the matrix forms for the gates $R_X(\theta), R_Y(\theta), R_Z(\theta)$.

13. Show that $XYX = -Y$. Also, show that $XR_y(\theta)X = R_y(-\theta)$.

14. Prove that for any unitary U, the operator $cU = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U$ is a Unitary.

15. Let A and B be unitaries then show that the operator

$$|0\rangle\langle 0| \otimes (I \otimes I) + |1\rangle\langle 1| \otimes (A \otimes B) = (|0\rangle\langle 0| \otimes (I \otimes I) + |1\rangle\langle 1| \otimes (A \otimes I))(|0\rangle\langle 0| \otimes (I \otimes I) + |1\rangle\langle 1| \otimes (I \otimes B))$$

16. Use the Grover-Rudolph state preparation procedure to prepare the state

$$|\psi\rangle = \sqrt{0.3}\,|00\rangle + \sqrt{0.4}\,|01\rangle + \sqrt{0.1}\,|10\rangle + \sqrt{0.2}\,|11\rangle$$

17. For a given unitary U, V is defined as $\sqrt{U}$. Verify the action of the following circuit as the ccU gate.