Due: February 6th, 2025
Date: February 20, 2025

HW3 (25 points)

This homework is experiment heavy! Start early.

Identities on Pauli and other operators:

$XYZ = iI$
$\text{Tr}(X) = \text{Tr}(Y) = \text{Tr}(Z) = 0$
$X^2 = Y^2 = Z^2 = I$
$HXH = Z, HYH = -Y, HZH = X$

Problem 1. *(Points :20)* Let $|\alpha\rangle = \frac{1}{\sqrt{3}}|010\rangle + \frac{2}{3}|101\rangle - \frac{\sqrt{2}}{3}|110\rangle$.

(a) (4 points) Design a quantum circuit to prepare $|\alpha\rangle$ from $|000\rangle$ using the Grover-Rudolph method (described in class). Additional ancillæ may be used, if necessary. Describe the steps used to design the circuit.

(b) (3 points) Implement it using Qiskit. Submit the qasm/python code.

Run your code using the statevector simulator for 100,500,1000,2000,3000,5000,10000 shots and QPU for 5000 shots. First, read the next questions to see what you need to do after running.

(c) (1 point) Suppose ρ is a pure state. Prove that $\sqrt{\rho} = \rho$.

(d) (1 point) Fidelity of two states ρ and σ is defined as $F(\rho, \sigma) = \left(\text{Tr}\left\{\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right\}\right)^2$. Prove that $F(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|) = |\langle\psi|\phi\rangle|^2$.

(e) (1 point) For the statevector simulator, plot the fidelity of $|\alpha\rangle$ and the state that is created with respect to the number of shots. This simulator will give you all the amplitudes with which you can easily compute fidelity.

It is not straight-forward to determine if a QPU created the state $|\alpha\rangle$ correctly; remember that you can only perform measurements on a QPU to probe its state (this problem is called *quantum state tomography*, and is interesting enough for an implementation/survey project). You will try a naive quantum tomography idea in the next few questions.

(f) (1 point) Suppose we have a Hermitian operator $\rho = c_1 I \otimes X \otimes X + c_2 X \otimes Z \otimes Y$. Prove that c_1 can be calculated from $\text{Tr}\{\rho(I \otimes X \otimes X\})$ and c_2 from $\text{Tr}\{\rho(X \otimes Z \otimes Y)\}$. You will find the above identities helpful.

(g) (1 point) Suppose we have a quantum circuit creating some 3-qubit state ρ. Explain how $\text{Tr}\{\rho(I \otimes X \otimes X\})$ can be computed with the help of measurements in the Pauli basis. Then, explain how to compute the same using only standard-basis measurements (a question in HW1 tells you how to simulate measurements in any basis using standard-basis measurements).

(h) (3 point) Denote $|\alpha\rangle\langle\alpha|$ by σ. We learnt in class that we can always write σ as a linear combination of tensor-product of 3 Pauli operators from $\{I, X, Y, Z\}$.

$$\sigma = \sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} c_{ijk} P_i \otimes P_j \otimes P_k,$$

where $P_0 = I, P_1 = X, P_2 = Y, P_3 = Z$. Compute the coefficients for σ using Pennylane [1] and submit it. Let's call the list of coefficients $\overline{C} = [c_{000}, c_{001}, \ldots c_{333}]$.

[1] https://docs.pennylane.ai/en/stable/code/api/pennylane.pauli_decompose.html

(i) (4 point) Estimate each c_{ijk} on the QPU using the ideas in questions (f) and (g). Call the list of estimated coefficients $\overline{D}$ in the similar ordering. Use 5000 shots for estimating probabilities on the QPU.

(j) (1 point) Submit a table with four columns ijk, $P_i \otimes P_j \otimes P_k$, c_{ijk} (actual coefficient), and d_{ijk} (estimated coefficients).

Problem 2. *(Points :5)* Consider two bits $b_0, b_1 \in \{0, 1\}$, and consider the operator $U = Z^{b_0} X^{b_1}$.

(a) (2 points) Express HUH using Pauli operators (and b_0, b_1).*Hint: You will find the above identities helpful.*

(b) (3 points) Consider the quantum teleportation protocol. Suppose, instead of the Bell state, Alice and Bob share the entangled state $(I \otimes H) |\beta_{00}\rangle$. Show how that protocol can be modified so that the state generated at Bob's end is $H |\psi\rangle$ where $|\psi\rangle$ denotes the state that Alice wanted to send. Show the modified circuit and all calculations. *Hint: Your answer to (a) may be useful here.*

This idea is used by some research labs to implement the Hadamard operator on some physical qubit even when the Hadamard operation cannot be implemented on that qubit due to hardware limitations. This, of course, requires that a special entangled state be prepared and shared but the state preparation and sharing challenge can be tackled differently.
