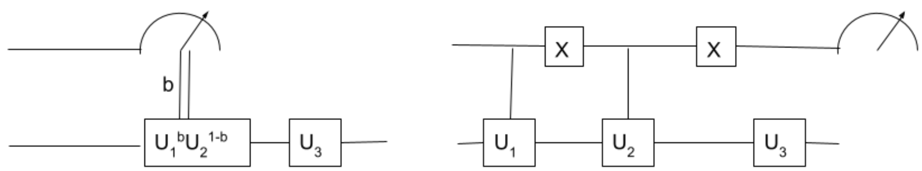
CSE622 (IQC), Introduction to Quantum Computing **Due: February 6th, 2025**
Winter 2025, IIIT-Delhi *Date: January 27, 2025*

HW2 (25 points)

Problem 1. *(Points :1+2+2=5)* Consider these two circuits. In the left one, a classically controlled gate is applied (just like in the quantum teleportation protocol). If the measurement outcome of the 1st qubit is 0, U_2 is applied on the 2nd qubit and if the outcome is 1, U_1 is applied. The measurement of the 1st qubit here is in the standard basis (but the general fact is true for any basis).



Let $|\psi\rangle$ denote the state of the 1st qubit and $|\phi\rangle$ denote the state of the 2nd qubit.
(a) Write down the density operator corresponding to the state of the 2nd qubit in the left circuit after applying the classically controlled gate.

(b) Write down the state of the right system *before* the measurement step.

(c) Write down the density operator corresponding to the state of the right system *after* the measurement step. *Hint:* This should match the answer of (a).

The above equivalence shows intermediate measurements can be effectively pushed to the end, even if there are quantum gates that are applied conditioned on the classical outcomes of the measurements.

Problem 2. *(Points :1+1+2+3+2+2+2=13)* A SWAP operator is a two register operator that swaps their states in the standard basis. That is,

$$|\phi\rangle |\psi\rangle \xrightarrow{SWAP} |\psi\rangle |\phi\rangle, \quad \phi, \psi \in \{0,1\}^n$$

I will use $SWAP_n$ to denote n-qubit swap operator.
(a) Write down the outer-product form of the single-qubit $SWAP_1$ operator.

(b) Write down the spectral decomposition of the $SWAP_1$ operator. You do not have to show calculations; you are permitted to use some external tools to compute the eigen-decomposition of an operator.

A controlled-SWAP operator is a very useful operator that applies the SWAP operator on *two* target registers only when a control qubit is set. We can express such an operator as

$$cSWAP = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes SWAP$$

Do you see how the operator, written in this form, captures its logic described above?

(c) Suppose I use $cSWAP_1$ as my observable to measure the 3-qubit GHZ state. What is the expected value of my measurement, denoted $\langle CSWAP_1\rangle$?

(d) Suppose we apply $cSWAP_n$ on three registers initialized to $|+\rangle |\phi\rangle |\psi\rangle$ in which the first qubit is the control register and the second two are targets. Now, measure only the first qubit in the Hadamard basis. Compute the probabilities of the different outcomes and the state of the remaining systems corresponding to the different outcomes. Express the probabilities in terms of $r = |\langle\psi|\phi\rangle|$.

(e) Create one quantum circuit to compute two sets of states, in two different registers: $|\psi\rangle = |\beta_{01}\rangle$ and $|\phi\rangle = |{+}{+}\rangle$. Use the technique in (e) on these registers as target. Plot the probability of $|0\rangle$ as you increase runs from $0 \ldots 10000$.

(f) Let ρ and σ denote the (mixed) states corresponding to the first qubits of $|\phi\rangle$ and $|\psi\rangle$, respectively. Compute $\mathrm{Tr}\{\rho\sigma\}$.

(g) Finally, perform the same experiment but only on the first qubits of $|\phi\rangle$ and $|\psi\rangle$.
Hint: Your answer should be converging to $\frac{1}{2}(1 + \mathrm{Tr}\{\rho\sigma\})$

For (e) and (g) submit the qasm code for the circuits and the plots.

Problem 3. *(Points :3)* Consider an arbitrary two-qubit state:

$$|\psi\rangle = \alpha_{00}\,|00\rangle + \alpha_{01}\,|01\rangle + \alpha_{10}\,|10\rangle + \alpha_{11}\,|11\rangle$$

Define the concurrence of such a state as $C_2 = 2|\alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10}|$. Show that ψ is an bipartite state if and only if $C_2 = 0$. We can define a state to be bipartite if it can be written as a tensor product of two single-qubit states. A state that is not bipartite is a said to be entangled.

Problem 4. *(Points :2+2=4)* Consider the Bell state $|\beta\rangle = \frac{1}{\sqrt{2}}[|01\rangle - |10\rangle]$. Let U be any single-qubit unitary operator.

(a) Prove that applying U on the first qubit (say, held by Alice) is completely equivalent to applying U^{-1} on the second qubit (say, held by Bob).
Hint: Thus, even if Alice is stranded on a classical island without a quantum computer, she could have U operated on her qubit by requesting Bob to apply U^{-1} on his qubit!

(b) Which of the other Bell states share this same property?
