CSE622 W24/Quantum Computing: Homework 1

Announced: 10th Jan, 2025 (25 points), Due date: 20th Jan, 2025, 11:59pm

USE BRA-KET notation as much as possible.

Upload scanned copies of your handwritten solutions or PDFs of digital solutions.

1. [6 points] Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ be some single-qubit state.

(a) [1 point] What is the distribution of outcomes if we measure $|\psi\rangle$ in the Hadamard basis?

(b) [1 point] What is the distribution of outcomes if we measure $|\psi\rangle$ in the standard basis after applying the Hadamard operator?

Create an IBMid account on IBM Quantum.

(c) [4 points] Write a python program that performs the following operation 100 times:

- Select two bits a and a' uniformly at random.
- Select a state $|\psi\rangle$ according to a by following the B92 protocol.
- Measure $|\psi\rangle$ in a basis that is chosen based on a' by following the B92 protocol. Use the idea in (a) and (b) to obtain the measurement outcome in the Hadamard basis since Qiskit does not allow directly measuring in any basis other than the standard basis.
- Draw a table in your homework with the following columns: a, a', shared key (if generated) or -1 (if round is discarded). Finally, write the shared key generated and its length.

2. [2 + 2 + 1 + 1 + 6 = 12 points] Consider the B92 protocol. Suppose, instead of sending the states as discussed in class, Alice decides to send $|\psi_0\rangle = \frac{1}{\sqrt{3}}|0\rangle + i\sqrt{\frac{2}{3}}|1\rangle$ and $|\phi_0\rangle = \frac{1}{2}|0\rangle - \sqrt{\frac{3}{4}}|1\rangle$. In this question you will analyse this protocol.

(a) Extend $|\psi_0\rangle$ to a basis $B_0 = \{|\psi_0\rangle, |\psi_1\rangle\}$. Show your calculations.

(b) Extend $|\phi_0\rangle$ to a basis $B_1 = \{|\phi_0\rangle, |\phi_1\rangle\}$. Show your calculations.

(c) Explain what happens when you measure $|\psi_0\rangle$ in B_0.

(d) Explain what happens when you measure $|\phi_0\rangle$ in B_1.

(e) Analyse the correctness of the B92 protocol, in the absence of Eve, if Alice sends $|\psi_0\rangle$ to encode $a = 0$ and $|\phi_0\rangle$ to encode $a = 1$, and Bob measures in B_0 if $a' = 0$ and in B_1 if $a' = 1$.

3. [1 + 3 + 3 = 7 points] A qudit is an element of the 4-dimensional Hilbert space, denoted $\mathcal{H}_4$. Some researchers are studying hardware implementations of qudits and exploring whether it is beneficial as compared to a qubit-based QPU.

(a) Write the vector forms of the standard basis of $\mathcal{H}_4$. We will call them $|0\rangle, |1\rangle, |2\rangle, |3\rangle$.

(b) Write a basis for $\mathcal{H}_4$ that includes the state $|+\rangle = \frac{1}{2}[|0\rangle + |1\rangle + |2\rangle + |3\rangle]$. For every basis state, all the amplitudes corresponding to the standard basis states must be $\frac{1}{2}$. Call the basis states $B = |{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle$. Prove they B is a basis.

(c) Express $|\psi\rangle = \sqrt{\frac{1}{5}}|0\rangle + \sqrt{\frac{2}{5}}|1\rangle + \sqrt{\frac{1}{5}}|2\rangle - \sqrt{\frac{1}{5}}|3\rangle$ in the B basis.