28 Feb, 2025 | CSE622 – Quantum Computing: Midsem Exam | *Out of 30 points*

120 minutes *Instr: Debajyoti Bera*

Common states: $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, $|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$.

Common gates: H maps $|0\rangle$ to $|+\rangle$ and $|1\rangle$ to $|-\rangle$, X maps $|0\rangle \to |1\rangle$ and $|1\rangle \to |0\rangle$, Z maps $|0\rangle \to |0\rangle$ and $|1\rangle \to -|1\rangle$, $XYZ = iI$ (compute Y from this identity), rotations gates $R_P(\theta)$ (where P is some Pauli operator) acts as $R_P(\theta) = e^{-iP\frac{\theta}{2}} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}P$, and CNOT maps $|a\rangle|b\rangle \to |a\rangle|a \oplus b\rangle$ for $a, b \in \{0, 1\}$.

Use identities: $HXH = Z, HYH = -Y$.
You should always provide explanation regarding correctness of your answers. Use of matrix-notation instead of bra-ket notation will **attract penalty.** Ignore the **COx, COy, . . .** tags associated with the questions – they are included due to statutory reasons. I designed the *-questions to be less straight-forward compared to the others.

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(1) [0 points] How was your preparation for this exam? Not sufficient / Average / Confident

(2) [5 points, CO1] Design a single-qubit state $|\psi\rangle$ that, upon measurement using a suitable observable, should collapse to $|+\rangle$ with probability 1/3 and $|-\rangle$ with probability 2/3; further, $\langle 0|\psi\rangle$ should have a non-zero imaginary component.

- (a) [1 point] Write down such a state in the standard-basis.
- (b) [1 point] What observable could have been used above? Write down the spectral decomposition of the observable.
- (c) [1 point] What are the outcomes if you measure the state using the observable in (b)? State the probabilities of the different outcomes and the corresponding post-measurement states.
- (d) [2 point] Draw a quantum circuit that prepares $|\psi\rangle$ from $|0\rangle$ and explain how it operates gate by gate.

(3) [1+1+2=4 points, CO1] Consider the following operator: $\rho = \frac{1}{3}|+\rangle\langle +| + c|0\rangle\langle 0|$.

- (a) [1 point] Suppose $c = \frac{2}{3}$. Is ρ a valid density operator? Explain.
- (b) [1 point] If ρ is a valid density operator, explain whether ρ represents a mixed state or a pure state. If ρ is not a valid density operator, suggest a value of c to make ρ a valid operator.
- (c) [*2 point] In (b), we settled on a valid ρ. Design a two-qubit state $|\psi\rangle$ such that $\mathrm{Tr}_2(|\psi\rangle\langle\psi|)$ equals ρ. Show the derivation of $\mathrm{Tr}_2(|\psi\rangle\langle\psi|)$.

(4) [2+1+1+1=5 points, CO1] Let E be some single-qubit observable.

- (a) Suppose we measure the first qubit of the Bell state $|\beta_{00}\rangle$ using E. Compute $\langle\beta_{00}|E \otimes I|\beta_{00}\rangle$ in terms of the eigenvalues of E.
- (b) Write down the outer-product form of some operator U such that $|\beta_{10}\rangle = U|\beta_{00}\rangle$.
- (c) Express the eigenvalues of $U^\dagger(E \otimes I)U$ in terms of the eigenvalues of E.
- (d) (*) Compute $\langle\beta_{10}|E \otimes I|\beta_{10}\rangle$ in terms of the eigenvalues of E. *Hint: Use (a), (b), (c).*

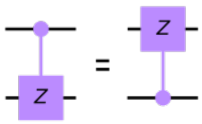
(5) [1+2+1=4 points, CO2, CO3] This is a question on the Deutsch-Jozsa algorithm.

- (a) Draw a circuit for the Deutsch-Jozsa's algorithm for a 2-bit function $f : \{0,1\}^2 \to \{0,1\}$. Your circuit can use either U_f or $\hat{U}_f$ (phase version of U_f).
- (b) Suppose the function satisfies $f(x_1x_2) = s_1x_1 \oplus s_2x_2$ for two secret bits s_1 and s_2. Write down the output state of the above circuit given a quantum oracle to such an f.
- (c) Suppose we measure the output state in the standard basis. Write down the probability of observing $|s_1s_2\rangle$.

(6) [3+1=4 points, CO2] Let $A = [a_1, a_2, \ldots a_n]$ be an integer array and T be an integer. (a) Explain how to construct a Hamiltonian H such that the ground state energy of H is 0 if and only if some elements can be chosen from A whose sum is T. (b) Show your approach on $A = [3, 11, 7]$ and $T = 10$.

(7) [1+2=3 points, CO2] (a) What is $QFT_8 \left|5\right\rangle$ where QFT_8 denotes the 3-qubit quantum Fourier transform operator? (*b) Show that the output state is not entangled, i.e., it can be written as a tensor-product of three single-qubit states.

(8) [1+2=3 points, CO3] (a) Prove that, unlike the CNOT gate, the control and the target qubits of a two-qubit control-Z gate can be arbitrarily chosen. That is, prove that these two gates are identical.



(b) Explain how to implement a control-Z gate using CNOT and any single-qubit gates.

(9) [*2 points, CO3] This question is about the Grover-Rudolph state preparation algorithm. Let $T(n)$ denote the number (or, an upper-bound on it) of gates required to prepare any n-qubit state. Derive the gate-complexity of the algorithm by first writing down a recurrence for $T(n)$ (along with an explanation of the same) and then solving it. You can assume that if an operator U can be implemented using k gates, then the operator control-U can be implemented using $2k$ gates.

(10) [0 points] How did you find the question paper? Easier than expected / Balanced / Harder than expected / Beyond my reach even on the best day!