3 May, 2025 | CSE622 – Quantum Computing: Endsem Exam | *Out of 70 points*

180 minutes | *Instr: Debajyoti Bera*

Common states: $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, $|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$.

Common gates: H maps $|0\rangle$ to $|+\rangle$ and $|1\rangle$ to $|-\rangle$, X maps $|0\rangle \to |1\rangle$ and $|1\rangle \to |0\rangle$, Z maps $|0\rangle \to |0\rangle$ and $|1\rangle \to -|1\rangle$, $XYZ = iI$ (compute Y from this identity), rotations gates $R_P(\theta)$ (where P is some Pauli operator) acts as $R_P(\theta) = e^{-iP\frac{\theta}{2}} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}P$.

Use of matrix-notation instead of bra-ket notation will **attract penalty.** Ignore the **COx, COy, . . .** tags associated with the questions – they are included due to statutory reasons.

Part-A (30 points)

(No explanation or derivation steps are required unless explicitly mentioned.)

(0) [0 points] **(a)** How was your preparation for this exam? Insufficient / Average / Confident
(b) In hindsight, would you have preferred notes shared by me as opposed to you taking notes in class? Y/N

(1) [2 points, CO1] Write down two single-qubit Hamiltonians H_1 and H_2 such that H_1 is also unitary but H_2 is not unitary. None of them can be the identity operator. Write the Hamiltonians in their outer-product form.

(2) [1 point, CO1] Write down an expression for the maximally mixed state in the basis $\{|+\rangle, |-\rangle\}$. *A maximally mixed state is a uniform mixture of standard basis states.*

(3) [1 point, CO1] Suppose both the qubits of $\frac{1}{\sqrt{2}}[|01\rangle - |10\rangle]$ are measured in the $\{|+i\rangle, |-i\rangle\}$ basis. State the different outcomes and their probabilities.

(4) [1 point, CO1] Write down the spectral decomposition of e^{-iXt} where t is some positive scalar.

(5) [1 point, CO1] Write down two single-qubit operators A and B (from the known set of such operators that we used in the class) such that $e^{i(A+B)} \neq e^{iA}e^{iB}$.

(6) [2 points, CO1] Calculate $Tr(0.3I + 0.5X + 0.7Y + 1.2Z)$.

(7) [1+2=3 points, CO3] **(a)** Consider the 2-bit function $f(00) = 0$, $f(01) = f(10) = f(11) = 1$. Write down a unitary operator U_f to access f; U_f should be expressed as a sum of outer-products in the standard basis.

(b) Now, suppose that an n-bit f is known to be either balanced or constant. Draw a quantum circuit using a similar unitary operator U_f to determine if f is balanced or constant. Clearly write the initial state of the qubits. Explain how the measurement output can be used to determine the answer.

(8) [2 points, CO1, CO3] Calculate $\|I - Z \otimes Z\|$ ($\|\cdot\|$ denotes the operator norm).

(9) [2 points, CO2] Write down a Hamiltonian whose ground state energy is equal to the optimal value of the following QUBO:

$$\vec{x}^T Q \vec{x} \text{ where } Q = \begin{bmatrix} 1 & -1 & 2 & -2 \\ -1 & 5 & 3 & 4 \\ 2 & 3 & -3 & -5 \\ -2 & 4 & -5 & 4 \end{bmatrix}$$

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(10) [1+2=3 points, CO2, CO3] A is an n-qubit quantum circuit. Let p denote the probability of observing a good state when $A|0^n\rangle$ is observed.
(i) Determine p such that we can run the amplitude amplification circuit on A to obtain a good state with certainty (without any error) such that exactly 7 calls to the Grover iterator are made.
(ii) Now, suppose that $p = 0.01$ and we run the amplitude amplification circuit by calling the Grover iterator the optimal number of times. What would be the error in this case? *This question may appear difficult to do without a trigonometric calculator; you are advised to use known approximations of trigonometric quantities and leave answers in terms of cos, sin, etc.*

(11) [1 point, CO3] Consider the 6-qubit state $|\psi\rangle = \frac{1}{8}[|0\rangle + |8\rangle + |16\rangle + |24\rangle + |32\rangle + |40\rangle + |48\rangle + |56\rangle]$. Write the state obtained after applying $QFT^\dagger$ on $|\psi\rangle$.

(12) [11 points, CO2] Suppose I want to use Grover's algorithm to search for "1" in a binary array $A = [0, 1, 0, 0, 0, 0, 1, 0]$. We want any one of the two corresponding indices.

(a) [1 point] Write down a unitary operator U_A to access the array A. You can represent U_A either in the outer-product form or as a mapping on a suitable basis and you can use either the state version or the phase version of the oracle.

(b) [1 point] How many qubits would be required for the quantum circuit for implementing Grover's algorithm.

(c) [1 point] Write down the expression for the Grover iterator G in terms of U_A and other operators.

(d) [2 points] Draw the quantum circuit to implement G and make it as detailed as possible.

(e) [1 point] How many times should the circuit call G to solve the problem with probability > 0.5? Write down the optimal number of calls.

(f) [1 point] Draw the entire quantum circuit in terms of U_A and other basic gates. If you do not know how many times G should be called, you can apply G two times.

(g) [1 point] What should be the initial state of all the qubits of the quantum circuit?

(h) [2 points] Write down the final state of the circuit drawn above.

(i) [1 point] Which projective measurement should be applied on the final state to solve the problem? Write down the set of projectors on the entire set of qubits.

Part-B (40 points)

(You need to explain the correctness of your answer. Show all relevant derivations.)

1. [7 points, CO3] Draw a quantum circuit to implement the following $k+1$-qubit operator: $\sum_{a=0}^{2^k-1} |a\rangle\langle a| \otimes H^a$. If you are unable to do this, you can solve it for the simpler case of $k = 2$ for at most 3 points.

7 points

- 1 - looks like a quantum circuit
- 1 - one controlled gate per qubit of first register
- 2 - circuit is correct
- 3 - analysis

3 points

- 1 - looks like a quantum circuit
- 1 - circuit is correct
- 1 - analysis

2. [10 points, CO3] We discussed in class that any n-qubit observable M can be written as a linear-combination of tensor-product of Pauli operators:

$$M = \sum_{j_1 \in \{0,1,2,3\}} \sum_{j_2 \in \{0,1,2,3\}} \cdots \sum_{j_n \in \{0,1,2,3\}} \beta_{j_1 j_2 \dots j_n} Pj_1 \otimes P_{j_2} \otimes P_{j_n}, \text{ where}$$

P_t is some Pauli operator from $\{I, X, Y, Z\}$ in which t indicates the operator: $t = 0$ stands for I, $t = 1$ stands for X, $t = 2$ stands for Y, and $t = 3$ stands for Z. For example, $H = \frac{1}{\sqrt{2}}P_1 + \frac{1}{\sqrt{2}}P_3$.
Explain how to obtain all the coefficients $\{\beta_{j_1 j_2 \dots j_n}\}$.
If you are unable to do this for a general M, find the decomposition of the following M in terms of I, X, Y, Z for at most 4 points:

$$M = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$$

3. [5 points, CO3] Suppose we have two circuits C_1 and C_2 each consisting of a sequential application of 10 gates: $C_1 = U_9 U_8 \cdots U_0$ and $C_2 = V_9 V_8 \cdots V_0$. Further, suppose $\|U_i - V_i\| \leq 0.01$, $\|U_i - I\| \leq 0.005$ and $\|V_i - I\| \leq 0.005$. Let p_1 and p_2 denotes the probabilities of observing a good state upon measuring $C_1|\psi\rangle$ and $C_2|\psi\rangle$, respectively. Relate p_1 and p_2 mathematically. Show all steps of your derivation.
If you are unable to do this, compute $\|C_1 - C_3\|$ where $C_3 = U_9 U_8 U_7 U_5 U_4 U_3 U_2 U_1 U_0$ for at most 2 points.

4. [5 points, CO2, CO3] Derive an Hamiltonian G and a scalar t such that $e^{iGt} = H$ (Hadamard operator) (there is no unique answer to this question).
   If you are unable to do it, derive the spectral decomposition of H (without explicitly computing the determinant of any matrix) for a maximum of 2 points (*hint:* $H^2 = I$ *should give you an idea about its eigenvalues.)*.

5. [7 points, CO2, CO3] Let U be an n-qubit unitary operator and $|\psi\rangle$ be some eigenstate of U; suppose we know that the eigenvalue corresponding to $|\psi\rangle$ can be represented by an m-bit phase. Draw the circuit for QPE on U and $|\psi\rangle$ and analyse the gate complexity.

6. [6 points, CO2, CO3] Consider the Hamiltonian $H = Z \otimes X$; let $U = e^{iHt}$ where t is any scalar. Show how to simulate $(e^{iHt})^k$ without any error. Draw the final circuit and explain why it implements the given operator without any error. You circuit may use ancillæ if required but they must be initialized to $|0\rangle$ and reset to $|0\rangle$, and it should be composed of gates such as CNOT, Toffoli, and any single-qubit gate.
   If you are unable to do this, you may solve it for $H = Z \otimes Z$ and $k = 1$ for a maximum of 3 points.