Worksheet 5
IQC 2024

1. Show that $tr(\rho^2) < 1$ for a purely mixed state of the form $\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$ where $p_i \notin \{0, 1\} \;\; \forall\, i$.

2. Which of the following are valid density operators?

   (a) $\begin{bmatrix} 1/4 & (1-i)/4 \\ (1-i)/4 & 3/4 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 4/9 & 2\sqrt{5}/9 \\ 2\sqrt{5}/9 & 5/9 \end{bmatrix}$

3. Give two examples of mixed states $\rho_1 = \sum_i p_i |\psi_i\rangle \langle\psi_i|$ and $\rho_2 = \sum_j q_j |\phi_j\rangle \langle\phi_j|$ such that $|\psi_i\rangle \neq |\phi_j\rangle$ for any i and j but $\rho_1 = \rho_2$.

4. Let $|a\rangle = \frac{1}{\sqrt{3}} |+\rangle + \sqrt{\frac{2}{3}} |-\rangle$ and $|b\rangle = \frac{2}{3} |+\rangle - \frac{\sqrt{5}}{9} |-\rangle$. Assume a mixture σ with 75% states prepared in $|a\rangle$ and 25% states prepared in $|b\rangle$.

   (a) Write down $\rho_a = |a\rangle\langle a|$ and $\rho_b = |b\rangle\langle b|$.

   (b) Write down the density matrix of σ.

   (c) On measuring σ in $\{|+\rangle, |-\rangle\}$ basis, what is the probability of obtaining the outcomes $|+\rangle$ and $|-\rangle$.

   (d) On measuring σ in $\{|0\rangle, |1\rangle\}$ basis, what is the probability of obtaining the outcomes $|0\rangle$ and $|1\rangle$.

5. Let ρ be a density operator of a two-qubit system AB. Show that $Tr_B\big((U \otimes Y)\rho(U^\dagger \otimes Y^\dagger)\big) = U Tr_B(\rho) U^\dagger$.

6. The expectation of an operator A with respect to a mixture of states ρ is given as $\langle A\rangle = Tr(\rho A)$. Compute $\langle X\rangle$ where $\rho = \begin{bmatrix} 1/3 & i/3 \\ -i/3 & 2/3 \end{bmatrix}$.

7. Say Alice and Bob share the two-qubit state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B - |1\rangle_A |1\rangle_B)$. Let ρ be the density matrix corresponding to this two-qubit state.

   (a) Compute the density matrix ρ_A for Alice by taking the partial trace with respect to Bob's basis.

   (b) Next, compute the density matrix ρ_B for Bob by taking the partial trace with respect to Alice's basis.

   (c) Show that both Alice and Bob share the same mixed state.