Worksheet 3
IQC 2024

14 Feb 2024

1. Consider the two qubit state $|a_1a_0\rangle$ for some $a_0, a_1 \in \{0,1\}$. Show that

$$(H \otimes H) |a_1a_0\rangle = \frac{1}{2} \sum_{y \in \{0,1\}^2} (-1)^{a \cdot y} |y\rangle .$$

   Can you show (possibly using induction) that

$$H^{\otimes n} |a\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{a \cdot y} |y\rangle$$

   for any $a \in \{0,1\}^n$?

2. Suppose that $f(00) = f(01) = 0, f(10) = f(11) = 1$. Apply the Deutsch-Josza algorithm and show that at least one of the first two qubits ends up as a 1.

3. Let $f : \{0,1\}^n \to \{0,1\}$ be a linear function, i.e, f can be defined as $f(x) = a \cdot x$ where $a \cdot x$ is the dot product modulo 2 of a and x. Given an oracle to such a function where a is not known, your task is to construct a quantum algorithm to find a. (Think Deutsch-Jozsa.)

4. We know that the Deutsch-Jozsa algorithm perfectly distinguishes between constant and balanced functions. Now, assume that we tricked the algorithm and gave it a function that is neither balanced nor constant. What would the output of the algorithm look like?

5. Show that the Grover iterator defined as

$$G = \mathcal{A}\big(2 \left|0^{\otimes n}\middle\rangle\middle\langle 0^{\otimes n}\right| - \mathbb{I}\big)\mathcal{A}^\dagger\ V$$

   is unitary, where A is a quantum algorithm, and V is an oracle that marks the good states (the phase oracle discussed in the class).

6. Let $x = x_0x_1x_2x_3 = 0100$ be a 4 bit input. Assume that x is given to you as an oracle. You need to search for an index i such that $x_i = 1$.

   (a) Write down the Grover iterator corresponding to one step of amplification for this problem.

   (b) Write the state of the amplification circuit after applying each operator in the Grover iterator for 1 iteration.

   (c) What is the success probability of obtaining an index i such that $x_i = 1$?

   (d) What is the state of the amplification circuit after applying the Grover iterator for another iteration?

   (e) Now, what is the success probability of obtaining an index i such that $x_i = 1$?

7. Say you are to perform an unordered search over a list of n items, all of which are unique.

   (a) What is the number of qubits required to perform Grover's search for this problem? What is the query complexity of this problem?

(b) (H) Now, assume that you only have S space (qubits) available where $1 \leq S \leq \lceil \log n \rceil$. Can you devise an algorithm to solve this search problem with just S qubits? What is the query complexity of this problem, given that the space is constrained to just S qubits?

8. Let $|\psi\rangle$ be a state and let O be a Hermitian operator. Show that the expectation value of measuring O on $|\psi\rangle$ is $\langle\psi| O |\psi\rangle$.

9. Assume the $p(y|x)$ follows normal distribution $\mathcal{N}(Wx + b, I)$. Show that maximizing $\log p(y|x)$ on data is equivalent to performing least squares regression, i.e., minimizing $\|y - (Wx + b)\|^2$.

10. Prove that for any Hermitian operator O, the smallest eigenvalue is $min_{|\psi\rangle} \langle\psi| O |\psi\rangle$ where the minimization is performed over all unit vectors.