Worksheet 2
IQC 2024

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1. Show that the tensor product of two unit norm vectors is also of unit norm.
2. Prove that the tensor product of two unitaries is unitary.
3. Let Alice and Bob share $|\phi_-\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$. Say, Alice measures in $\{|b_0\rangle, |b_1\rangle\}$ basis. What would the state Bob obtains when Alice observes $|b_0\rangle$ on measurement? What would it be if Alice observes $|b_1\rangle$?
4. Show that for any arbitrary 2×2 unitary U, the equation $(U \otimes U)|\beta_{11}\rangle = |\beta_{11}\rangle$ holds up to a global phase.
5. Let $|\psi\rangle = \cos\theta|0\rangle + e^{i\phi}\sin\theta|1\rangle$. Let U be a 2×2 operator such that $U|0\rangle = |\psi\rangle$ and $U|1\rangle = -e^{i\phi}\sin\theta|0\rangle + \cos\theta|1\rangle$.
   (a) Is U a valid single-qubit unitary? Explain why or why not.
   (b) Can you give two example states $|\psi_1\rangle$ and $|\psi_2\rangle$ such that $\langle\psi|\psi_1\rangle = \langle\psi|\psi_2\rangle = 0$.
6. Let $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ be a function that takes two inputs each of size n and outputs an one bit output. Define an operator U such that

$$U|x\rangle|y\rangle|b\rangle|0\rangle = |x\rangle|y\rangle|b \oplus f(x,y)\rangle|0\rangle.$$

   Show that U is unitary.
7. Say, Alice and Bob are two parties, and Charlie is a mutual friend. Now, Charlie has a pair of entangled qubits shared with Alice. Incidentally, Charlie also has a pair of entangled qubits shared with Bob. Now, using these, can you create a pair of entangled qubits that are shared between Alice and Bob?
8. Construct a quantum circuit that on the state $|000\rangle$ creates the state $|GHZ_+\rangle = \frac{1}{\sqrt{2}}\big(|000\rangle + |111\rangle\big)$. Can you generalize this circuit to obtain a circuit that creates the n-qubit state $\frac{1}{\sqrt{2}}\big(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}\big)$ where $|0\rangle^{\otimes n}$ is the n-qubit all zero state and $|1\rangle^{\otimes n}$ is the n-qubit all one state?
9. Recall that during the teleportation protocol, Alice and Bob use the entangled state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$. Depending on the classical bits received by Bob from Alice, Bob applies X and Z gates to obtain the state intended to be teleported. Now, if Alice and Bob were to use the state $|\beta_{01}\rangle, |\beta_{10}\rangle$ or $|\beta_{11}\rangle$, when would Bob have to perform on his qubit to obtain the intended state?