Worksheet 1
IQC 2024

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Note: The questions are on page 1, and the hints for some questions are on page 2.

1. Find the inner product of the following states:
   (a) $|u\rangle = \sqrt{\frac{3}{5}}|0\rangle + \sqrt{\frac{2}{5}}|1\rangle$ and $|v\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
   (b) $|u\rangle = |+\rangle$ and $|v\rangle = \frac{8}{9}|0\rangle + \frac{i}{9}|1\rangle$.
   (c) $|u\rangle = \omega|0\rangle + \omega^2|1\rangle$ and $|v\rangle = |-\rangle$ where $1, \omega, \omega^2$ are the cube roots of unity.
2. Find the norm of the following:
   (a) $|u\rangle = \left(1 - \sqrt{3/4}\right)|0\rangle + \sqrt{i/4}|1\rangle$
   (b) $|u\rangle = \omega|0\rangle + \omega^2|1\rangle$
   (c) $|u\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
3. Which of the following is/are valid quantum states?
   (a) $|\psi\rangle = \frac{i}{\sqrt{2}}|+\rangle + \sqrt{2}|-\rangle$
   (b) $|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$
   (c) $|\psi\rangle = \frac{9}{10}|+\rangle + \frac{2}{10}|-\rangle$
4. For each invalid quantum state in the above question, modify the amplitude of one of the basis elements to obtain a valid quantum state.
5. Express the given states in $\{|+\rangle, |-\rangle\}$ basis.
   (a) $|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$
   (b) $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle$
   (c) $|\psi\rangle = -i\frac{\sqrt{15}}{4}|0\rangle + \frac{1}{4}|1\rangle$
6. Give 3 examples of single-qubit basis. How many more such bases can be found?
7. Recall that Hermitian matrices are matrices that are self-adjoint, i.e., $A = A^\dagger$. Is a product of two Hermitian matrices also Hermitian? What if the matrices are commutative?
8. We call the $\{|0\rangle, |1\rangle\}$ the standard basis, computational basis or Z-basis and $\{|+\rangle, |-\rangle\}$ as the Hadamard basis or X-basis. Can you reason why they might be called Z and X basis, respectively? Given this, if the Pauli-Y matrix is defined as $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, what would be the Y-basis?
9. For each state in Question 5, express the state in Y-basis.
10. Which of the following is/are valid projectors?
    (a) $P = \frac{1}{2}|+\rangle\langle+| - \frac{1}{2}|-\rangle\langle-|$
    (b) $P = \frac{1}{2}\Big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\Big)$
    (c) $P = \frac{1}{\sqrt{2}}\Big(|0\rangle\langle 0| + i|0\rangle\langle 1|\Big)$

Hints:

1. $IP(u, v) = u^{\dagger}v.$
3. Norm?
6. Hermitian $\implies A^{\dagger} = A$; Commutative $\implies A \cdot B = B \cdot A$
8. Eigen...?
10. $P^2 = P$