# HW4 (25 points)

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**Problem 1.** *(Points :1+1+2+1+2+4=11)* Consider the following problem of generating an examination time-table. Suppose that the DOAA office has to schedule $T$ exams (denoted $E^1 \dots E^T$) in exactly $T$ slots (denotes $S_1 \dots S_T$). There is a list of registered students for each exam (say, the students taking exam $E^i$ are denoted $s_1^i s_2^1 \dots$), and no student should be assigned to different exams in the same slot. Furthermore, the office wants to ensure that the overall free-time the students get *between* the exams is as large as possible.

One way to model this problem is to use the following decision variables: $X_{pa}$ which is set to 1 (representing True) if exam $p$ is scheduled in the slot $a$, and 0 (representing False) otherwise. Define *constants* $D_{ab}$ to be the "free time" available between time-slot $a$ and time-slot $b$ (e.g., suppose $a$ is Monday 9:00-11:00am and $b$ is Monday 3:00-5:00pm; then $D_{ab}$ is 4 hours); naturally, $D_{ab}$ is 0 if $b$ is earlier than $a$.

**(a)** Write down an equation that captures the constraint that each exam must be assigned to exactly slot.

**(b)** Write down an equation that captures the constraint that each slot must be allotted to exactly one exam.

Suppose a student is appearing for only one exam. Note that such a student imposes absolutely no restriction on the scheduling of that exam, i.e., scheduling that exam in no slot would violate any constraint related to that student.

Next, suppose a student will take only two exams – $E^p$ and $E^q$. Here is a generic expression that captures the free-time the student will get between these two exams: $\sum_{a=1}^{T} \sum_{b=1, b \neq a}^{T} D_{ab} X_{pa} X_{qb}$ which simply sums across all possibilities of slots for the two exams.

**(c)** Based on the above characterization, write a function that indicates the sum total of all between-exam free-times across all students. You may find these constants handy: $n_{pq}$ denotes the number of students who are taking both $E^p$ and $E^q$.

**(d)** Write an objective function for an unconstrained version of the above optimization problem.

**(d)** Write down a Hamiltonian $H$ whose ground state energy equals the optimal value (free-time) of the above problem.

**(e)** Solve the above optimization problem for the instance given in the attached datasheet [1]. You may use either DWave's QUBO solver or any quantum-simulator based VQE solver [2]. The VQE approach may be trickier, so you will get 2 BONUS marks for following this approach. You have to submit a PDF of your notebook for this part.

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**Problem 2.** *(Points :2)* Prove that $\|U - V\|_{op} = \|U^\dagger - V^\dagger\|_{op}$ for unitary operators $U$ and $V$.

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**Problem 3.** *(Points :1+1+1+3+1+2+1+2=12)* Consider the unordered search problem over $N = 2^n$ elements with a unique solution. Let $x$ denote the index of the unique solution. We want to analyse the performance of quantum annealing for solving this problem. Towards that purpose, consider these Hamiltonians $H_P = I - |x\rangle\langle x|$ and $H_D = I - |\psi\rangle\langle\psi|$.

**(a)** Describe the eigenstates, along with their corresponding eigenvalues, of $H_P$.

**(b)** State the ground state of $H_D$ and the corresponding energy.

*Hint: Both the above questions can be solved using inspection and the properties of the operators, i.e., you can avoid analysing large matrices.*

**(c)** Consider a time-varying mixture of those Hamiltonians: $H(s) = (1-s)H_D + sH_p$. Write down the matrix representation of $H(s)$ in the space spanned by $|x\rangle$ and $|x^\perp\rangle$ which is the uniform superposition of all standard basis states other than $|x\rangle$.

**(d)** Your matrix in (c), denoted $M(s)$ will have small dimensions if you got it right. Now, compute the eigenvalues of $M(s)$.

**(e)** Define $\Delta(s)$ as the *gap/difference* among the two ground state and the excited state energies. Plot $\Delta(s)$ against $s$ for $s = 0 \to 1$, assuming $n = 20$. You can use any plotting software – submit only the plot image/PDF.

**(f)** What is the minimum value of $\Delta(s)$ and for what value of $s$? I am looking for a general analytic expression here. The answers will be either constants or functions of $n$.

**(g)** What can you say about the time to be used to annealing based on your answer for (f)?

**(h)** Draw a quantum circuit to implement $U = e^{-iH_P \Delta t}$ for a small $\Delta t$. Explain how your circuit implements this operator.

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[1] Soon to be uploaded.

[2] You cannot use any library that takes as input a Hamiltonian and automatically returns the optimum value; you have to write the code for calculating the expected value of the Hamiltonian by taking multiple measurements. If in doubt, ask the TA.