HW2 (25 points)

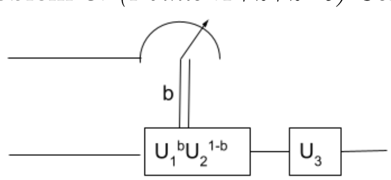
Problem 1. *(Points :5)* The objective of this question is to implement the operator $C-C-H$.

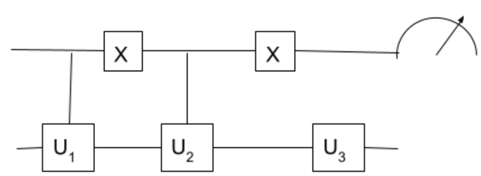
1. *(Points :2)* Consider the (single-qubit) operator G such that $G^2 = H$. Express G as $U(\theta, \phi, \lambda)$ according to the specification of QISKIT U-gate [1]. Similarly express $G^\dagger$ too.
2. *(Points :1)* Read Theorem-4.1 and Corollary-4.2 of Mike & Ike (Nielsen-Chuang) and solve Exercise 4.12 for the G gate: *Give A, B, C and α for the G gate.*
3. *(Points :2)* Use the ideas presented in Figures 4.5, 4.6, and 4.8 of Mike & Ike to implement the $C-C-H$ operator. Draw the final circuit that should use only CNOT and single-qubit gates. Specify the operations of the single-qubit gates in any unambiguous manner.

Problem 2. *(Points :5)* In this question you will design a two-party cooperative remote state preparation protocol. Three parties Alice, Bob, and Charlie share the 3-qubit entangled GHZ state. Alice and Bob want to help Charlie prepare the state $|\psi\rangle = \cos\theta\,|0\rangle + e^{\iota\phi}\sin\theta\,|1\rangle$; however, Alice only knows θ and Bob only knows ϕ. The objective is to design a protocol that involves the following: Alice measuring in some basis and sending the (classical) outcome to Charlie, Bob measuring in some basis and sending the (classical) outcome to Charlie, and Charlie taking some action based on the bits he received. I will give you the first step.

1. *(Points :1)* Alice measures in the following basis and sends the outcome to Charlie. $|A_1\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$ and $|A_2\rangle = -\sin\theta\,|0\rangle + \cos\theta\,|1\rangle$. Describe the measurement statistics and the states of the system after the measurement.
2. *(Points :2)* Identify a suitable basis in which Bob should measure. Describe the measurement statistics and the states of the system after measurement in that basis. *Hint: Carefully assess the similarity of the collapsed states in the earlier part and $|\psi\rangle$.*
3. *(Points :2)* Describe the operations of Charlie based on the outcomes he received from Alice and Bob so that he can (re)create $|\psi\rangle$.

Problem 3. *(Points :1+2+2=5)* Consider these two circuits.





The measurement of the 1st qubit here is in the standard basis (but the general fact is true for any basis). Show first that the probabilities of observing the standard basis states are identical. Then, show that the states of the 2nd qubit are identical for both the circuits when the first qubit is observed as $|0\rangle$. Finally, show the same when the first qubt is observed as $|1\rangle$.

The above equivalence shows intermediate measurements can be effectively pushed to the end, even if there are quantum gates that are applied conditioned on the classical outcomes of the measurements.

[1] https://docs.quantum.ibm.com/api/qiskit/qiskit.circuit.library.UGate

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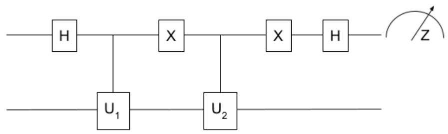
Problem 4. *(Points :2+1+2=5)* Consider the circuit below.



Suppose $U_1 \left|0\right\rangle = \left|\psi\right\rangle$ and $U_2 \left|0\right\rangle = \left|\phi\right\rangle$. Derive the probability of observing $\left|0\right\rangle$ if the circuit is given $\left|0\right\rangle \left|0\right\rangle$ as input states; relate the probability to $Real(\left\langle\phi\right| \psi\rangle)$. Now, modify the circuit such that the probability of the same can be related to $Imaginary(\left\langle\phi\right| \psi\rangle)$; show the calculation of the probability.

Problem 5. *(Points :1+2+1+1=5)* This is an implementation exercise on QISKIT. Consider a 4-bit function f such that $f(x) = 1$ for all x. Implement U_f. Consider another 4-bit function g such that g is balanced. Write down the truth table of g. Implement U_g. Now implement the Deutsch-Jozsa algorithm on QISKIT, and run the algorithm 100 times each with both the oracles separately. Report the success probabilities for both the oracles.
