CSE622 W24/Quantum Computing: Homework 1

Announced: 16th Jan, 2024 (25 points), Due date: 26th Jan, 2024, 11:59pm

USE BRA-KET notation as much as possible.

Upload scanned copies of your handwritten solutions or PDFs of digital solutions.

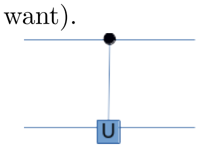
1. (2 points) Show that $XYX = -Y$ and use this to prove that $XR_y(\theta)X = R_y(-\theta)$. Avoid matrix multiplication, and instead, rely on outer-product or exponential representations of operators.

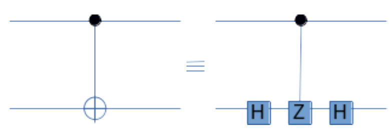
2. (2 points) Describe an operator G (either as action on basis states or as a linear combination of linear operators) such that $G^2 = H$.

3. (3 points) Let $\{|1\rangle, \ldots, |d\rangle\}$ denote the standard basis for a d-dimensional Hilbert space and let U be a unirary operator. Therefore, we get another basis $\mathcal{B}$ with states like $|\alpha_i\rangle = U|i\rangle$. Show that measuring any state $|\psi\rangle$ in the $\mathcal{B}$ basis is same as first applying $U^\dagger$ to $|\psi\rangle$ and then measuring in the standard basis (same implies that the probability of different possibilities are identical, and so are the post-measurement states). This will show that if we have an apparatus for measuring only in the standard basis, then we can effectively measure in any other basis.

4. (3 points) Let U be any 1-qubit unitary operator. We can define *controlled-U* as the following operator: $C - U = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U$. The $C - U$ gate is depicted using the left figure below.

Now, consider the circuit identity shown in the right figure. Prove that the left and the right circuits are identical. You can prove this either by showing that the corresponding operators are same (when written in the outer-product form), or by showing that the circuits produce identical states on any set of basis states (you can choose any basis that you want).





Hint: An H-gate only on the second gate can be represented by the operator $I \otimes H$. Thus, the right-most circuit can be seen as applying three operators in succession, from left to right.

5. (2 points) Prove that $e^{i\theta A} = \cos\theta I + i \sin\theta A$ if $A^2 = I$.

6. (2+1=3 points) Consider a project measurement implemented using an observable M. Consider some state $|\psi\rangle$, and for some θ, let $|\phi\rangle = e^{i\theta}|\psi\rangle$. (a) Show that the measurement outcomes and statistics would be completely identical whether $|\psi\rangle$ or $|\phi\rangle$ is measured. This shows that global phases do not have any observable impact, and therefore, can be ignored for all practical purpose. (b) Describe two states $|\alpha\rangle = a_0|0\rangle + a_1|1\rangle$ and $|\beta\rangle = a_0|0\rangle + e^{i\theta}a_1|1\rangle$ which can be distinguished by some measurement with zero error. This shows that local phase differences do have observable impact.

7. (4 points) Derive the probability that a round is discarded in the B92 protocol in the presence of Eve. For simplicity, assume that Eve's only action is to intercept the state sent by Alice, measure it using the X or the Z observable (chosen equally likely), recording the outcome, and sending the state thereafter to Bob. As discussed in the lecture, this probability should be sufficiently away from 3/4 for Alice and Bob to determine the presence of Eve.

8. (6 points) Create an account on IBM Quantum. Then perform the following experiments on the simulator. Submit a Python notebook containing the code, pictures of quantum circuit(s), explanation of the experiment process, and discussion of results.

(a) (2 points) Create a state that is either $|1\rangle$ or $|+\rangle$ with equal probability. Now, implement the first algorithm discussed in class that simply measures it in the standard basis and tries to predict the state. Compute the probability of error of this algorithm by running your algorithm 1000 times.

(b) (4 points) Do the same as above, but by using the optimal algorithm discussed in class.