27 Feb, 2024 | CSE622 – Quantum Computing: Midsem exam | *Out of 30 points*

120 minutes | *Instr: Debajyoti Bera*

You should always provide explanation regarding correctness of your answers, especially, for algorithms, protocols & circuits and their complexity, wherever relevant.

I and your TA are of the opinion that the question paper is a bit lengthy – about 1 question; I am keeping it that way to allow you more options. The maximum marks you can score from this test is 35; however, I will consider the total as 30 for computing your final score. Score of 6 or below is a matter of serious concern, and 20 or above is good.

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(1) [1 point] Consider a 3 qubit system. Write down an eigenstate of the Hamiltonian $H = X_1 + X_2 + X_3$, where X_i denotes the operator that applies X on the i-th qubit and nothing on the others. Also, state the eigenvalue corresponding to this eigenstate. *Hint: First find out or recall the eigenstates & the corresponding eigenvalues of the X operator – you will get 0.5 if you get this correct.*

(2) [2 points] Which of the following is/are valid projectors? Briefly explain. *If you are unable to solve these, you can state the necessary and sufficient conditions for an operator to be a projector – you will get 0.5 if those are correct.*

(a) $P_1 = \frac{1}{2}\Big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\Big)$

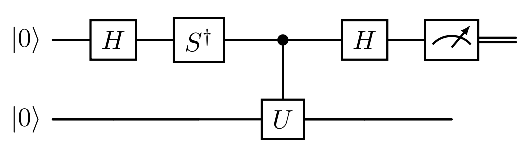
(b) $P_2 = \frac{1}{\sqrt{2}}\Big(|0\rangle\langle 0| + i\,|0\rangle\langle 1|\Big)$

(3) [2 points] Let $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ be a function that takes two inputs each of size n and outputs an one bit output. Define an operator U such that

$$U|x\rangle|y\rangle|b\rangle = |x\rangle|y\rangle|b \oplus f(x,y)\rangle .$$

Prove that U is unitary. *Even if you are unable to solve this, if you can explain the necessary and sufficient conditions that U should follow, you will get 0.5 if you are correct.*

(4) [2+1=3 points] Chart the evolution of the following quantum circuit with the initial state $|0,0\rangle$ for some arbitrary unitary U – show the state after each gate. What is the probability of observing $|0\rangle$ in the first qubit? Your answer will be in terms of U, rather, in terms of $\langle 0|U|0\rangle$. If you are unable to solve this for a generic U, you can assume that $U = H$ – you will get at most 2 if you are correct.



Even if you are unable to move past the first H gate, if you can write down the matrices of all the single-qubit gates, you will get 0.5 if you are correct.

(5) [5 points] We used the fact that a state-query oracle $|x\rangle|b\rangle \overset{U_f}{\to} |x\rangle|b \oplus f(x)\rangle$ can be used to implement a phase-query oracle $|x\rangle \overset{\hat{U}_f}{\to} (-1)^{f(x)}|x\rangle$. Give a quantum circuit to do the reverse, i.e., given a phase oracle $\hat{U}_f$ for an n-bit Boolean function $f()$, we want to implement its state-query version. For this, you may use other gates, ancilla, and also controlled-version of the phase-oracle gate $(c - U_f)$. For full points, make sure your additional qubits are returned to their initial states at the end of the circuit. A good first step would be to implement a special case $|x\rangle|0\rangle \overset{U_f}{\to} |x\rangle|f(x)\rangle$ – you will get (up to) 3.5 points if you manage to do this successfully. Whatever solution you propose, show the evolution of the states after each layer of gates.

(6) [2+1+2+2+1+3=11 points] Let x be a 4 bit string with exactly 2 bits set to 1. Assume that x can be queried using an oracle U_x. The task is to search for an index i such that $x_i = 1$.

(a) The naive way is to use an operator A to generate a uniform superposition of all indexes, and then measure it in the standard basis. This method can be repeated several times to increase the chance of success. Derive an upper bound on the probability that we will require 3 or more repetitions to find a good i.

(b) Now, let's use quantum techniques. Explain how you will implement A.

(c) Write down the Grover iterator corresponding to this problem as a composition of several operators each of which should be completely described.

(d) Write down the state of the system after 1 application of the Grover iterator. You need not show any derivation.

(e) Show that this approach is not error-free, i.e., if we measure after 1 iteration, there is a non-zero probability of observing an index i such that $x_i = 0$. Calculate this probability.

(f) Explain how to modify A so that 1 application of the Grover iterator is sufficient to obtain a good i with 100% probability. Show relevant calculations to justify the correctness of your modification.

(7) [2+3+3=8 points] (a) First, write down all the Bell states in the standard basis. (b) The objective of this part is to show that any Bell state can be converted to any other Bell state by apply single-qubit operators <u>only on the first qubit</u>. To simplify your answer, choose any Bell state of your choice and then, for all the other Bell states, explain how to convert the former to the latter by applying single-qubit operations only on the first qubit. (c) Now, consider a scenario where two parties Alice and Bob situated far apart share the former Bell state. Alice wants to inform two classical bits to Bob. Write down a protocol that allows Alice to do so by physically sending just one qubit; explain what Alice does, what Bob does, in what order, and how many times. *Hint: Alice, after some local operations, will send her qubit to Bob.*

(8) [2 points] Describe a quantum algorithm to compute kernel entries $K_{i,j}$ for any quantum feature map U. You may assume that you have access to a family of unitary operators $\{U(x)\}$ that creates a high-dimensional state $U(x)\left|0^m\right\rangle$ corresponding to a low-dimensional feature vector x.

Specifically, describe a quantum circuit (number of qubits, their initial value, operations), how to measure its output state (in which basis or specify the observable), and how to process the measurement outcomes.

(9) [0.5 points] How was your preparation for this exam? Not sufficient / Average / Confident

(10) [0.5 points] How did you find the question paper? Easier than expected / Balanced / Harder than expected / Beyond my reach even on the best day!