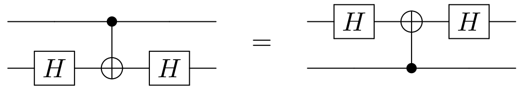
HW2 (25 points)

Problem 1. *(Points :2)* Describe the action of CNOT on all possible combinations of $|+\rangle$ and $|-\rangle$ states as input followed by measurement of both qubits using the X operator. Describe the possible post-measurement states, along with their probabilities, for input. Can you summarize your observation about the CNOT gates when used in this manner?

Problem 2. *(Points :1+1+2=4 points)* Show that $|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \times Z = I \otimes |0\rangle\langle 0| + Z \otimes |1\rangle\langle 1|$. Draw a circuit for the above operator and explain its semantics. Now, use the above observation along with the identity $Z = HXH$ to prove the equivalence shown in the following figure.



Problem 3. Alice and Bob would like to share the entangled state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Unfortunately, they do not initially share any entanglement, but fortunately, they have a mutual friend, Charlie. Alice shares a copy of $|\beta_{00}\rangle$ with Charlie, and Bob also shares a copy of $|\beta_{00}\rangle$ with Charlie.

1. *(Points :2)* Write the initial state using Dirac notation in the computational basis, with the first qubit belonging to Alice, the second and third qubits belonging to Charlie, and the fourth qubit belonging to Bob.
2. *(Points :4)* Suppose Charlie performs a Bell measurement on his two qubits (one of which is entangled with Alice and the other of which is entangled with Bob). For each possible measurement outcome, give the probability with which it occurs and the resulting post-measurement state for Alice and Bob.
3. *(Points :3)* Describe a protocol whereby Charlie sends a classical message to Alice, and Alice processes her quantum state, such that Alice and Bob share the state $|\beta_{00}\rangle$ at the end of the protocol.

Problem 4. Start with your birthdate, say D, and month, denoted M. Compute $A = (D \times M \mod 254) + 1$. Now write A as an 8-bit binary string $s = s_7 s_6 s_5 s_4 s_3 s_2 s_1 s_0$ (s_i is the ith LSB). Write down a Boolean function $f : \{0,1\}^3 \to \{0,1\}$ with the mapping: $f(000) = s0$, $f(001) = s1$, $f(010) = s_2$, $f(011) = s_3$, and so on.

1. *(Points :3+2=5)* Design a U_f operator that implements the $f()$ above in the following manner.

   $$U_f |b1, b2, b3\rangle |c\rangle = |b1, b2, b3\rangle |c \oplus f(b1, b2, b3)\rangle \quad \text{for } b1, b2, b3, c \in \{0,1\}$$

   Implement U_f using a sequence of CCC-NOT and X gates (essentially implement the logic: if input state is 00, output state is f(00), if input state is 001, output state is f(001), ...). Then, decompose each CCC-NOT gate using Toffoli and CNOT gates with the help of ancilla (use *clean computation*). Finally, use the techniques discussed in class to implement the Toffoli gates by first writing down a V such that $V^2 = X$. Finally, verify your implementation of U_f using QISKIT on some backend. Submit the picture of the circuit and the errors for each of the 16 inputs.
2. *(Points :3)* Theoretically derive the final state after the action of U_f on the following superposition: $\frac{1}{1/\sqrt{2}}[|000\rangle|-\rangle + |111\rangle|+\rangle]$. Explain the observations if we measure the 1st qubit of the final state using the X operator.
3. *(Points :2)* Run an experiment for the task in (b) and report your observations (report only the probabilities – there is no way to observe the post-measurement state in QISKIT).
