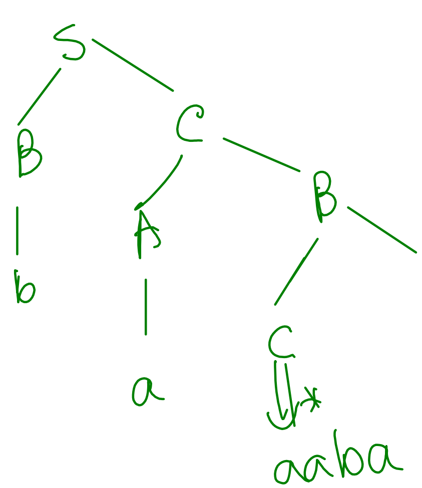
CSE322 Theory of Computation (L13)

Today

Pumping Lemma for PDA

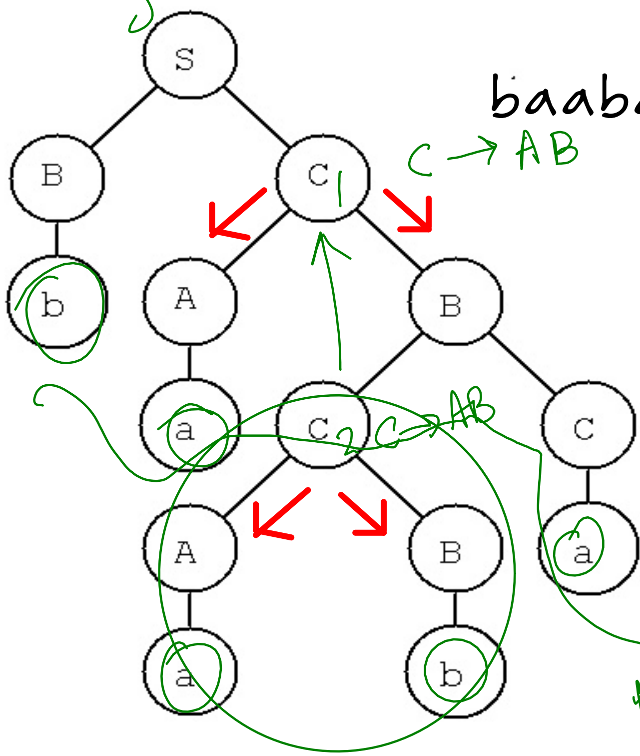


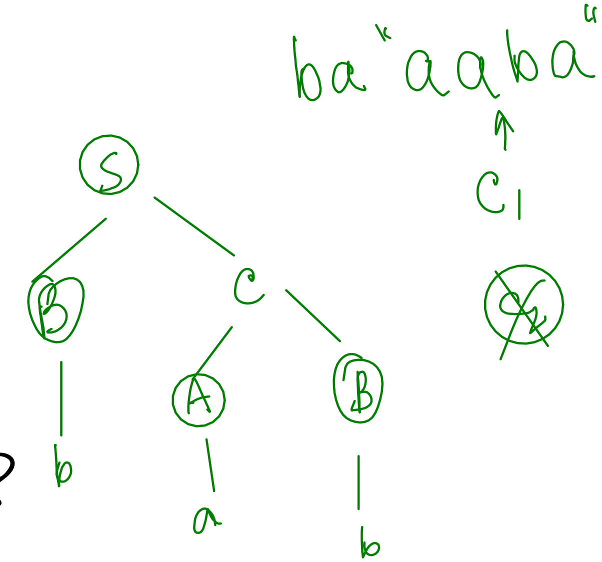
$\Rightarrow$ b aaa baa

Pumping lemma: Motivation

$S \overset{*}{\Rightarrow} baCa \Rightarrow S \overset{*}{\Rightarrow} baaba$

$C \overset{*}{\Rightarrow} ab$





Q: Is bab in L?

Q: What is the maximum number of leaf nodes of a tree with height h where b is the maximum number of children a node can have?

$S \overset{*}{\Rightarrow} bC$

$baCa \Rightarrow baABa \Rightarrow baaBa \Rightarrow baaba$

baaaba

$A \to aBaBcC \Rightarrow$ length=6

height: # edges to farthest leaf

b: maximum length of any rule

If height <= h, length of string <= b^h

If length of string >= b^h+1, height >= h+1

=> $\boxed{\text{Long strings cannot have short parse trees.}}$

Pumping Lemma

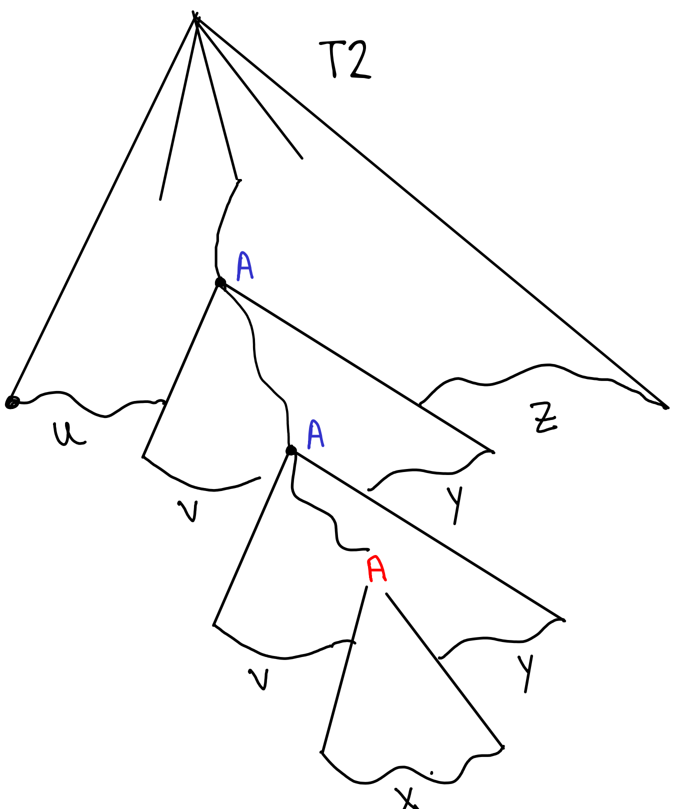
$p = b^v + 1$

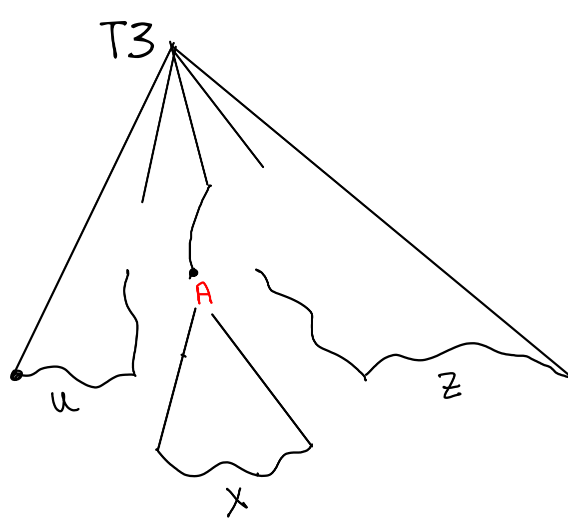
$p = b^{v+1} \geq b^v + 1$

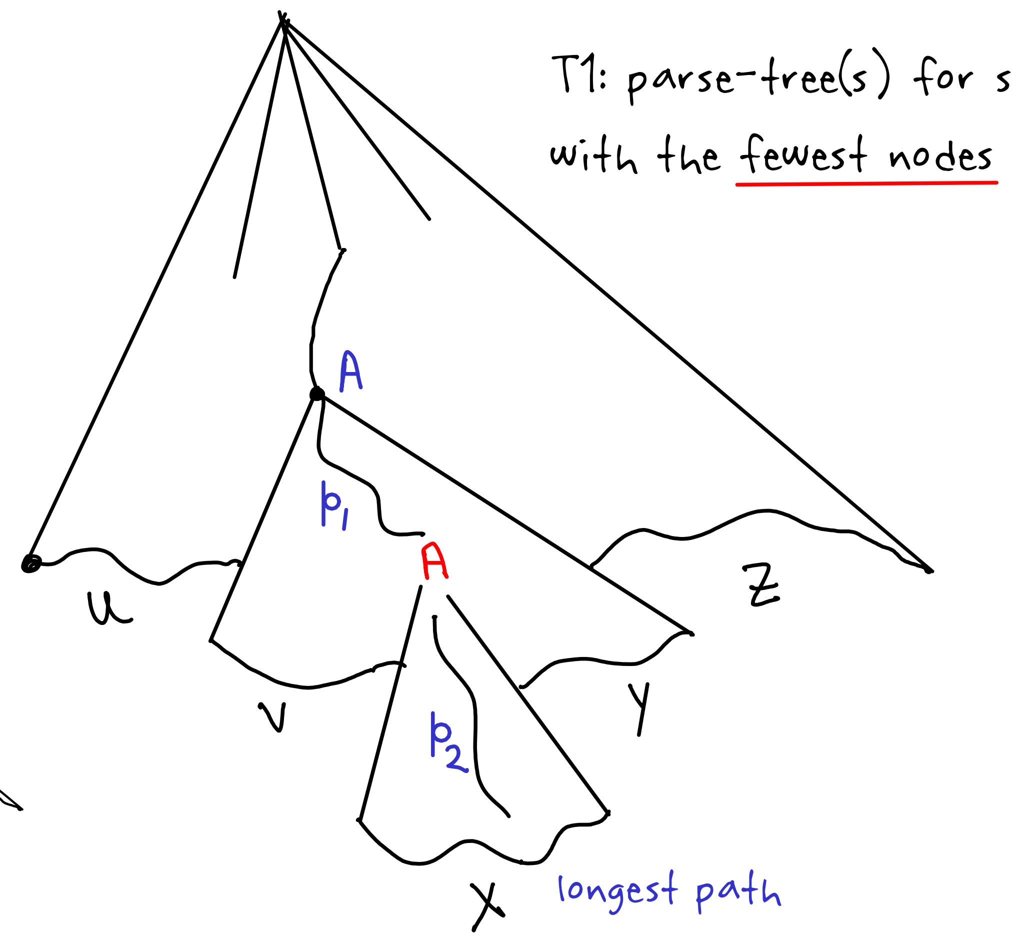
For CFL L, any string s in L of large length ($|s| >= p$) ?

can be divided into s = uvxyz s.t. → Does not say how to divide

* uxz, uvxyz, uvvxyyz, uvvvxyyyz, ... are in L ✓ ✓
* at least one of v and y is non-empty ✓ → o/w T3 generates w but with fewer nodes
* $|vxy| <= p$ ✓

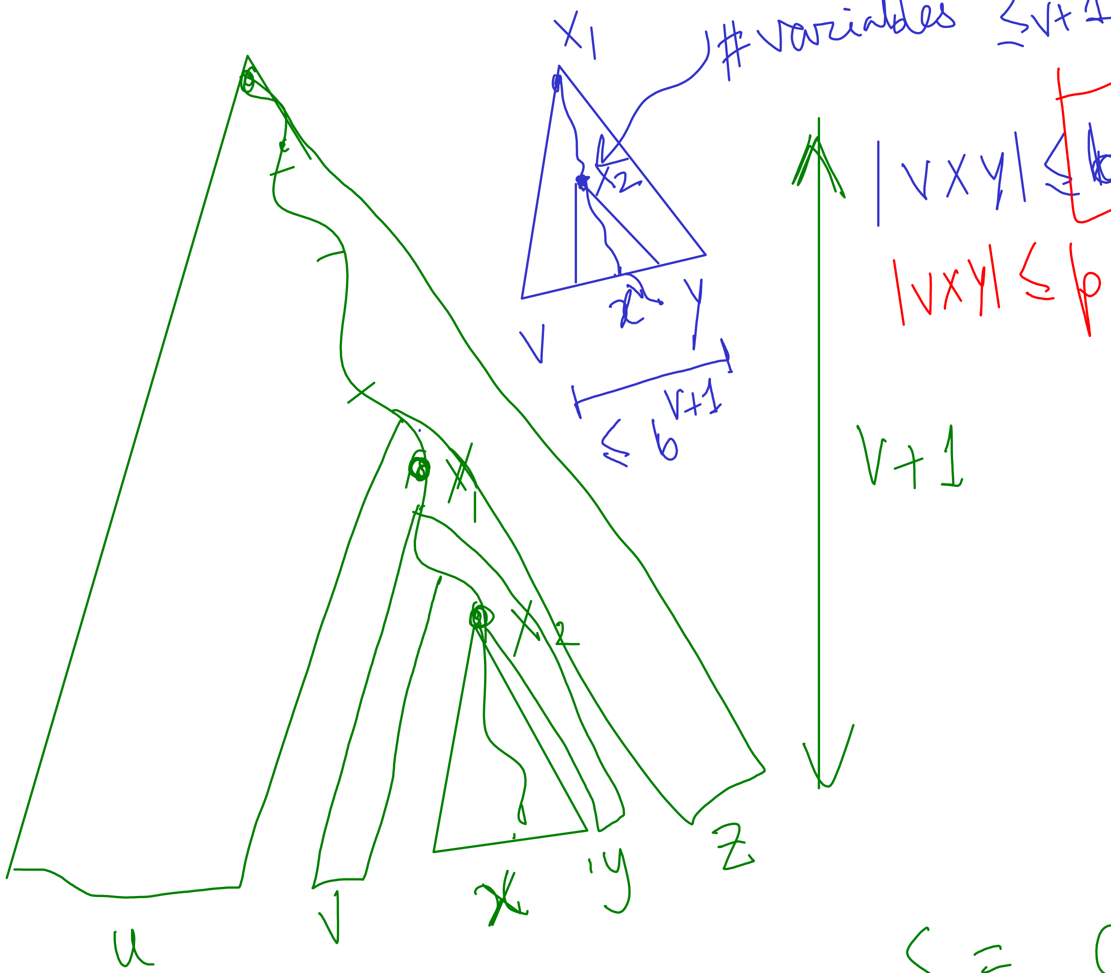






A is first repeated variable on the longest path from bottom: relate |vxy| with height of A

some variable must repeat if |s| >= .... ?



$|vxy| \leq \boxed{b^{V+1}}$

$|vxy| \leq p$

if $p = b^V + 1$?

$|vxy| \leq p$?

V : # variables

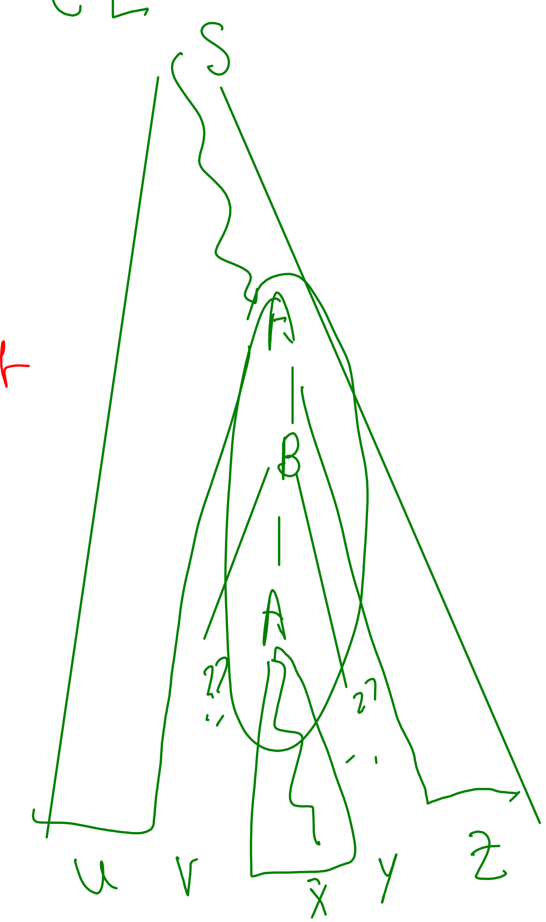
If s has length $\geq \boxed{b^V + 1}$, then ht of any parse tree for $s \geq V+1$

$s = uvxyz$

$\exists$ parsetrees for $\underbrace{uv^i x y^i z}_{\in L}$, $i \geq 0$

$s = \underbrace{aa}_{u}\,\underbrace{bb}_{v=\varepsilon\ \ x}\,\underbrace{cc}_{y=\varepsilon\ \ z}$.

$A \to \cdots$



$|vy| > 0$

$A \to B \quad B \to A$

both v, y cannot be empty

Shortest parse tree for $s \Rightarrow |vy| > 0$

$S \Rightarrow \cdots\ aaAbba \Rightarrow aaBbba \Rightarrow \cancel{aa}Abba$

$\Rightarrow \cdots$

$L = \{a^i b^j c^k : 0 <= i <= j <= k\}$ Show L is not regular.

Let L be CFL and p be the pumping length. $s = a^p b^p c^p = uvxyz$.
$\in L$

$\leftarrow p \rightarrow$ | $\leftarrow p \rightarrow$ | $\leftarrow p \rightarrow$

aaaaa....aaaa | bbbb......bbbb | cccc.....ccccc

vxy
pump up
#a

vxy
pump up
either
#a > #c
or #b > #c

vxy
pump up
#b

vxy!
pump down

vxy!
pump down
#c

$L = \{ww\}$

Let L be CFL and p be the pumping length.

$\overbrace{000...0000}^{p}\ 1\ \overbrace{000...0000}^{p}\ 1$

$\overbrace{000...0000}^{p}\ \overbrace{1111....11111}^{p}\ \overbrace{000...0000}^{p}\ \overbrace{111......11111}^{p}$

$s = ww$

vxy

vxy

vxy

vxy

vxy

vxy

vxy

Avoid too many cases

$|s| = 4p$

w' must end at 1

w' ends with 0

$s' = \ \ 0^p 1^p = w'w'$

w''

$0.5p$

$\leq p$

$4p+1 \leq |s'| \leq 5p$

$2p \leq |w'| \leq 2.5p$

w''

$w = 0^p 1^p$

Need to argue that pumped strings are not in L

* uvvxyyz would not be form w'.w'

Use: |vxy| <= p

Show: If vxy in first half, then after pumping once, second half starts with 1

Show: If vxy is in the middle pumping down doesn't work.

$0^p 1^a / 0^b 1^p \neq ww$

either $a < p$, $b < p$

$\neq w'w'$ for some w'

$L = \{$string over a,b,c s.t. $\#a < \#b$ & $\#a < \#c\}$

Let L be CFL and p be the pumping length.

$$\overbrace{aaa\ldots aaa}^{p}\ \overbrace{bbb\ldots bbbb}^{p+1}\ \overbrace{ccc\ldots cccc}^{p+1}$$

$a^p b^{p+1} c^{p+1}$

Case 1. vy contains a

Then vy cannot contain c.

Case 2. vy does not contain a

Then vy must contain either b or c.

Closure properties

L1 L2

G1

S1 -> AB

S1 -> BC

A -> BA | a

B -> CC | b

C -> AB | c

G2

S2 -> aAb | e

A -> aAbA | a

Union : L(G1) U L(G2)

S → S1 | S2

G1 … G2 …

Concatenation : L(G1).L(G2)

S → S1 · S2

Kleene Star : L(G1)*

S → ε | S . S1

Reversal of L(G2)

S → bAa | ε

A → AbAa | a

Exercise

Prove these using PDAs.

S ⇒ S S1 ⇒ S1 ⇒ …

$\{a^n b^n : n \geq 0\}$

Closure under intersection

$L = \{a^n b^n c^n : n \geq 0\}$ not CFL.

AnBnCn = { a^n b^n c^n : n >= 0} is not a CFL (prove using Pumping Lemma).

Write AnBnCn as intersection of two CFLs.

$L_1 = \{a^n b^n c^m : n \geq 0, m \geq 0\}$

$= \underbrace{\{a^n b^n : n \geq 0\}}_{CFL} \cdot \underbrace{\{c^m : m \geq 0\}}_{CFL}$

Closure under complement

Suppose CFLs are closed under complement. Now, arrive at a contradiction.

$L1$: CFL $\rightarrow \overline{\overline{L1} \cup \overline{L2}} = L1 \cap L2$

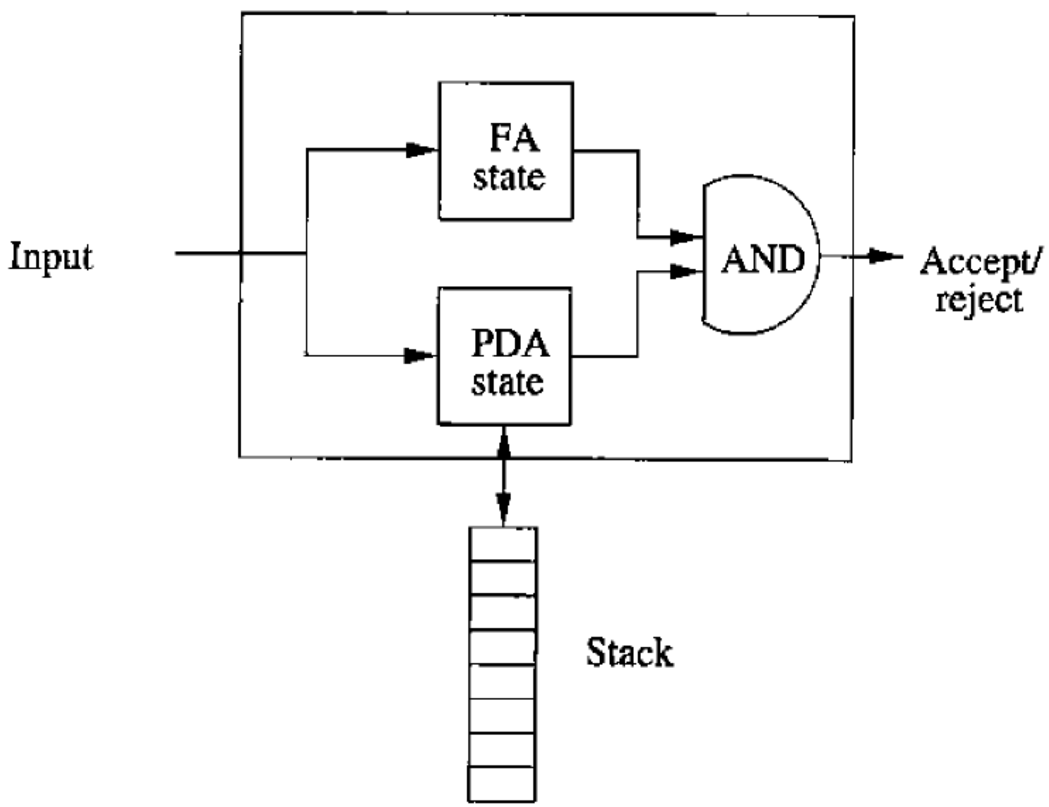
$L2$: CFL

$L2 = \{a^m b^n c^n : n \geq 0, m \geq 0\} \rightarrow$ CFL

$L1 \cap L2 = \{a^n b^n c^n : n \geq 0\}$.

CFLs and Intersection operation

CFL not closed under intersection !



CFL AND REG is CFL.

$\delta(\ldots)$

Show that $\{ a^n b^n c^n : n >= 0 \}$ is not a CFL using PL.

Show that ...

$\{w$ over $\{a,b,c\} : \#a(w) = \#b(w) = \#c(w) \}$ is not CFL

L = above language

L1 = a*b*c*

Prove that L intersect L1 is not CFL.

Then prove that L is not CFL.