# CSE322 Theory of Computation (L5)

## Recap of last lecture

## Today

NFA to DFA

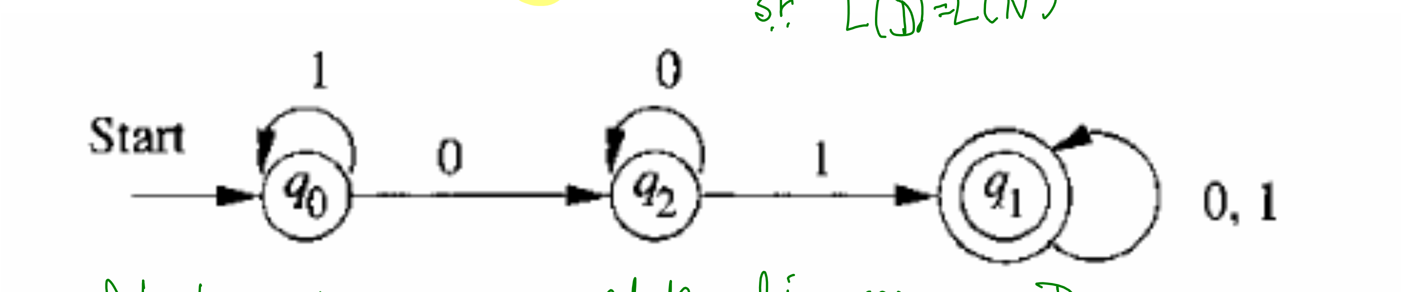
NFA proof of correctness

# DFA to NFA

Given DFA $M = \langle Q_D, \Sigma_D, \delta_D, q_{0D}, F_D \rangle$

construct equiv. NFA $N = \langle Q_N, \Sigma_N, \delta_N, q_{0N}, F_N \rangle$

s.t. $L(D) = L(N)$




N has the same state diagram as D.

$Q_N = Q_D$ ✓

$\Sigma_N = \Sigma_D$ ✓

$q_{0N} = q_{0D}$ ✓

$F_N = F_D$ ✓

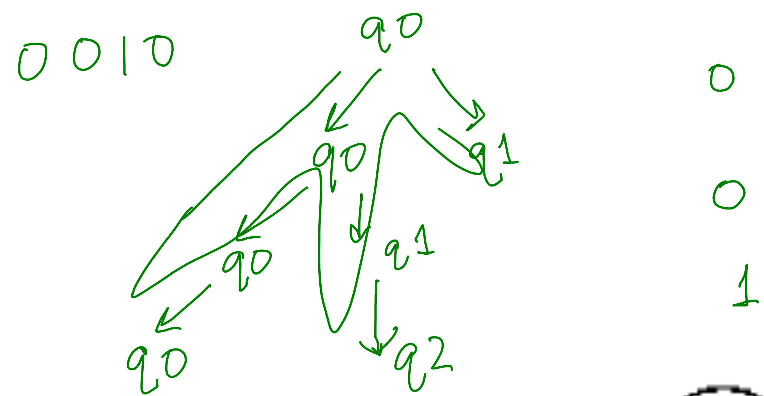
$\forall q \in Q_N, \forall s \in \Sigma_N$, $\delta_N(q, s) = \{ \delta_D(q, s) \}$

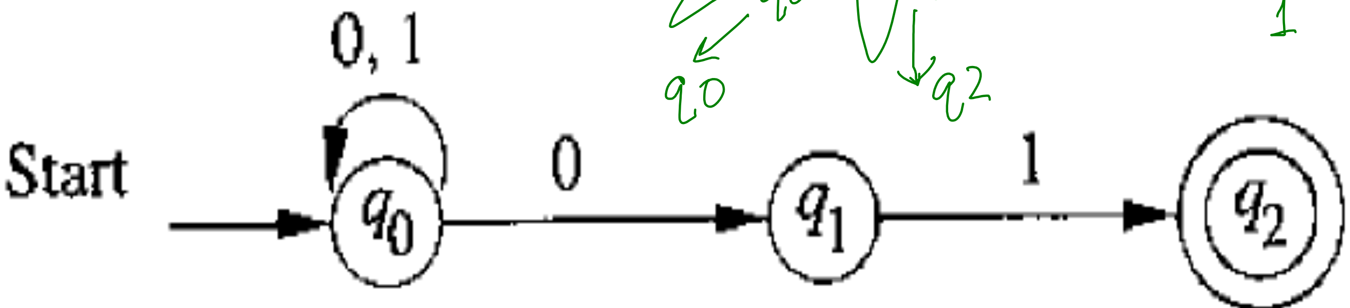
Given any DFA D, there exists/we can construct an NFA N s.t. $L(D) = L(N)$.

Given NFA N, construct DFA D s.t. L(D)=L(N).

# NFA to DFA (subset method)

(First, assume there is no ε-transition)




|  | 0 | 1 |
|---|---|---|
| $\rightarrow q_0$ | $\{q_0, q_1\}$ | $\{q_0\}$ |
| $q_1$ | $\emptyset$ | $\{q_2\}$ |
| $*q_2$ | $\emptyset$ | $\emptyset$ |

$\{q_0\} \xrightarrow{0} \{q_0, q_1\} \xrightarrow{0} \{q_0, q_1\} \xrightarrow{1} \{q_0, q_2\} \rightarrow \cdots$

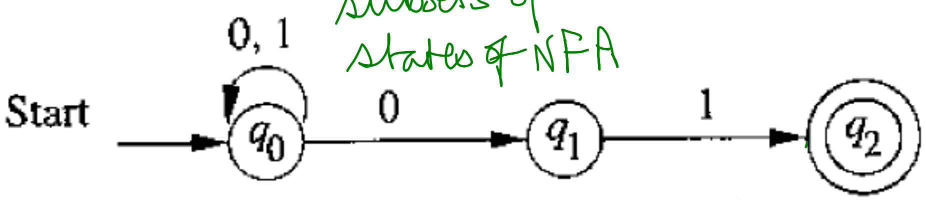
# NFA to DFA (subset method)

$NFA = \langle QN, \Sigma, \delta N, \delta 0N, FN \rangle$

$DFA = \langle \underline{\wp(QN)}, \Sigma, \delta D, \{q0N\}, FD = \{ Q' \subseteq QN \text{ s.t. } Q' \cap FN \neq \emptyset \} \rangle$

subsets of states of NFA

a state of DFA




| | 0 | 1 |
|---|---|---|
| $\rightarrow q_0$ | $\{q_0, q_1\}$ | $\{q_0\}$ |
| $q_1$ | $\emptyset$ | $\{q_2\}$ |
| $*q_2$ | $\emptyset$ | $\emptyset$ |

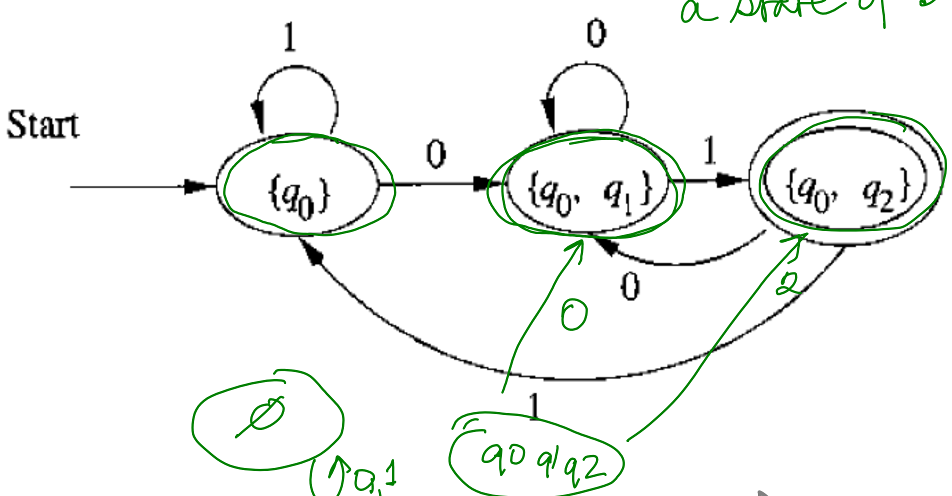
$D : q0 \xrightarrow{1} q0$

$D : \{q0, q1, q2\} \xrightarrow{1} \{q0, q2\}$

a state of DFA

$$\delta D(Q' \subseteq QN, a \in \Sigma) = \bigcup_{q \in Q'} \delta N(q, a)$$

$\cancel{Q' \in QN}$




(not showing unreachable transitions)

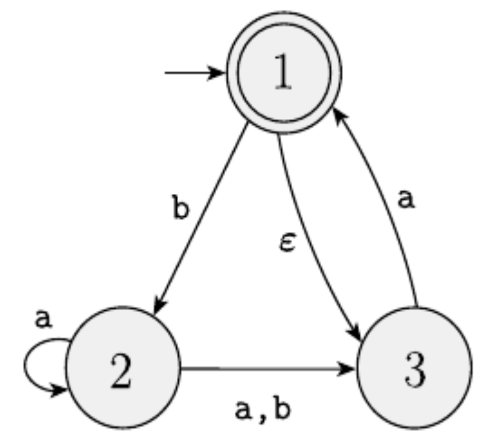
| | 0 | 1 |
|---|---|---|
| $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $\rightarrow \{q_0\}$ | $\{q_0, q_1\}$ | $\{q_0\}$ |
| $\{q_1\}$ | $\emptyset$ | $\{q_2\}$ |
| $*\{q_2\}$ | $\emptyset$ | $\emptyset$ |
| $\{q_0, q_1\}$ | $\{q_0, q_1\}$ | $\{q_0, q_2\}$ |
| $*\{q_0, q_2\}$ | $\{q_0, q_1\}$ | $\{q_0\}$ |
| $*\{q_1, q_2\}$ | $\emptyset$ | $\{q_2\}$ |
| $*\{q_0, q_1, q_2\}$ | $\{q_0, q_1\}$ | $\{q_0, q_2\}$ |

# ε-NFA to DFA (subset meth.)

Given ε-NFA N

construct equivalent DFA M

$\rightarrow \{1,3\}$




## ε-Closure

$E(q) = \{ q' \mid q \overset{\text{0 or more } \varepsilon\text{-rules}}{\rightsquigarrow} q' \}$

$E(\{q1, q2, \dots, qk\}) = ?$

$E(1) = \{1,3\}$

$E(2) = \{2\}$

$E(3) = \{3\}$

$E(1,3) = \{1,3\}$

$E(2,3) = \{2,3\}$

$E(1,2) = \{1,2,3\}$

...

$E(1,2,3) =$

$E(\{\}) = \{\}$

| $\delta_{NFA}$ | a | b | ε |
|---|---|---|---|
| $\rightarrow 1^*$ | ∅ | 2 | 3 |
| 2 | 2,3 | 3 | ∅ |
| 3 | 1 | ∅ | ∅ |

$1 \xrightarrow{a} 3,4$ $\quad E(1)$

$\downarrow\varepsilon$ 5,4 $\quad \searrow\varepsilon$ $7 \xrightarrow{\varepsilon} 9$, $\downarrow$ 10

$1 \xrightarrow{a} 3,4,5,7,9,10$

$7 \xrightarrow{\varepsilon} 9$

abab $\varepsilon_{q7}$ $\varepsilon_{q8}$ $\varepsilon_{q9}$ $\varepsilon$ $q10$

DFA: $\{3\} \xrightarrow{a} E(1) = \{1,3\}$

$D = \langle \underset{\text{same}}{Q_D}, \Sigma, \delta_D, \underset{\text{a state of DFA}}{E(q_{0N})}, F_D \rangle$

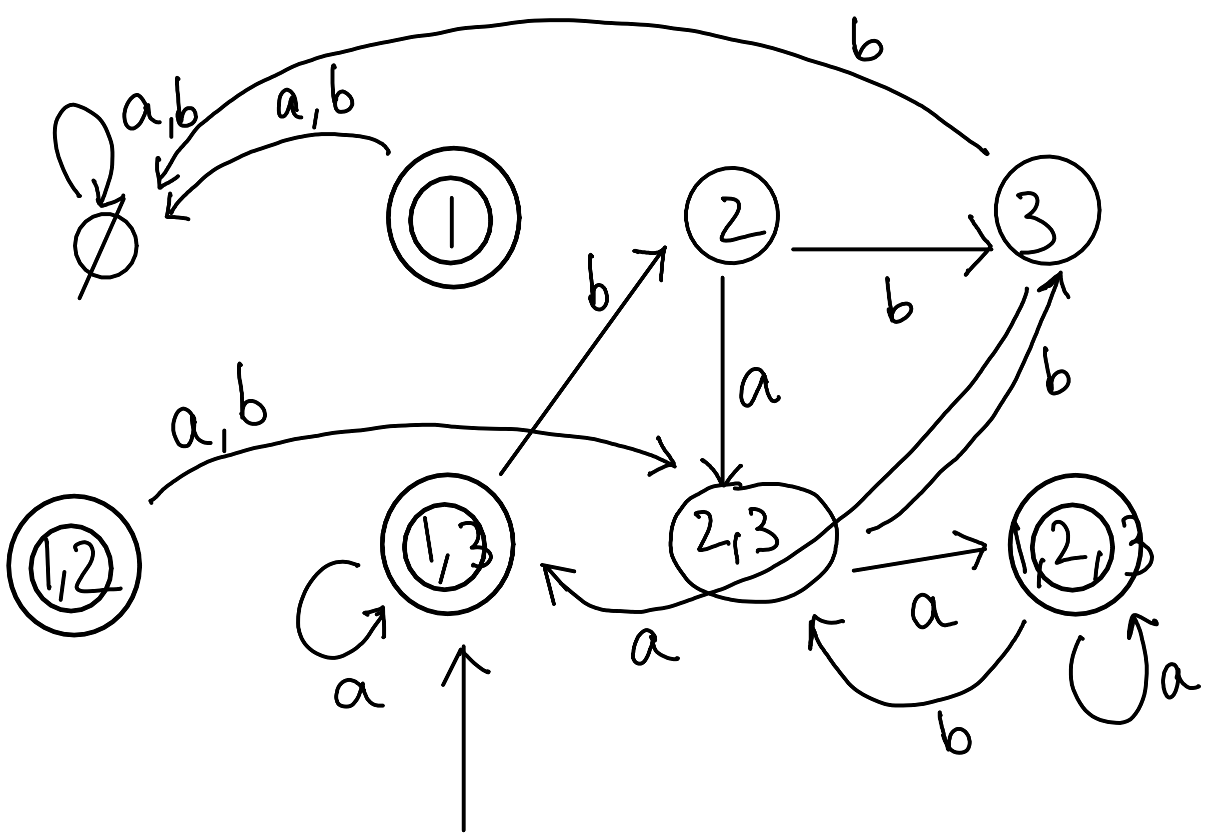
$\quad F_D = \{ Q' \subseteq Q_N \text{ s.t. } Q' \cap F_N \neq \emptyset \}$ (same)

$\delta_D(Q' \subseteq Q_N, a \in \Sigma) = \bigcup_{q \in Q'} E(\delta_N(q,a))$

$\cancel{Q' \in Q_N}$

For NFA, $\hat{\delta}(q, \omega)$ = set of states reached from state $q$ after reading string $\omega$

For $\varepsilon$-NFA, $\hat{\delta}(q, \omega)$ = like above, but also allowing $\varepsilon$-transitions




## Cross-check

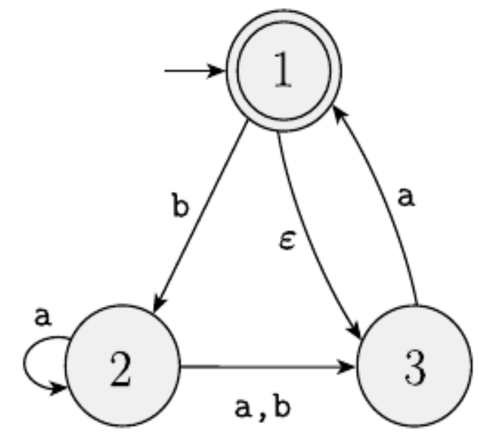
$\hat{\delta}(\text{start}, \varepsilon)$ → accept
$\hat{\delta}(\text{start}, a)$ → accept
$\hat{\delta}(\text{start}, baaa\cdots a)$ → accept
$\hat{\delta}(\text{start}, baa\cdots ba)$ → accept

$\hat{\delta}(\text{start}, b)$ → not accept
$\hat{\delta}(\text{start}, baa\cdots b)$ → not accept

# $\varepsilon$-NFA to DFA (subset meth.)

Given $\varepsilon$-NFA N

construct equivalent DFA M




**E-Closure** $E(q) = \{ q' \mid q \overset{\text{0 or more } \varepsilon\text{-rules}}{\rightsquigarrow} q' \}$

$E(1) = \{1, 3\}$  $E(2) = \{2\}$  $E(3) = \{3\}$  $E(\emptyset) = \emptyset$

$E(\delta_N(3, b) = \emptyset) = \emptyset$

| $\delta_{NFA}$ | a | b | $\varepsilon$ |
|---|---|---|---|
| $\rightarrow 1^*$ | $\emptyset$ | 2 | 3 |
| 2 | 2,3 | 3 | $\emptyset$ |
| 3 | 1 | $\emptyset$ | $\emptyset$ |

DFA states:

{ }, {1}, {2}, {3},
{1,2}, {1,3}, {2,3},
{1,2,3}

{2} → set of states of NFA
→ a state of DFA

| $\delta_{DFA}$ | a | b |
|---|---|---|
| $\{\emptyset\}$ | $E(\emptyset) = \emptyset$ | $E(\emptyset) = \emptyset$ |
| $*1$ | $E(\emptyset) = \emptyset$ | $E(\emptyset) = \emptyset$ |
| 2 | $E(2,3) = 2,3$ | $E(3) = 3$ |
| $\{3\}$ | $E(1) = 1,3$ | $\boxed{E(\emptyset) = \emptyset}$ |
| $*1,2$ | $E(2,3) = 2,3$ | $E(2,3) = 2,3$ |
| $\rightarrow *1,3$ | $E(1) = 1,3$ | $E(2) = 2$ |
| 2,3 | $E(1,2,3) = 1,2,3$ | $E(3) = 3$ |
| $*1,2,3$ | $E(1,2,3) = 1,2,3$ | $E(2,3) = 2,3$ |

# NFA to DFA (subset method)

$N = \langle Q, \Sigma, \delta_0^N, q_0^N, F_N \rangle \Rightarrow D = \langle \mathcal{P}(Q), \Sigma, \delta_0^D, E(\{q_0^N\}), \{Q' \subseteq Q : Q' \cap F_N \neq \phi\} \rangle$

$\forall Q' \subseteq Q,$ $\forall a \in \Sigma$ $\quad \delta_0^D(Q' \subseteq Q, a) = \bigcup_{q \in Q'} E(\delta_N(q, a))$

Claim :- $L(N) = L(D)$.

Ignore $\varepsilon$-NFA (for simplicity)

**Proof of $L(N) = L(D)$:** Define using extended transition functions

$\omega \in L(N)$ iff $\omega \in L(D)$

$F_D = \{Q' \subseteq Q : Q' \cap F_N \neq \phi\}$

$\omega \in L(D) \leftrightarrow$ $\boxed{\hat{\delta}_D(q_0^D, \omega)}^{\gamma} \in F_D \Leftrightarrow \boxed{\hat{\delta}_D(\{q_0^N\}, \omega)} \cap F_N \neq \emptyset$ // By lemma.

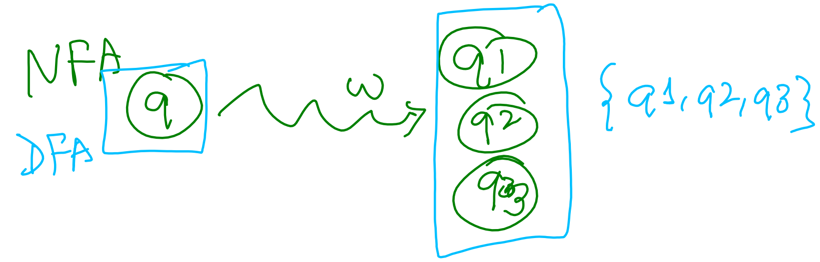
$q_0^D = \{q_0^N\}$

$\omega \in L(N) \longleftrightarrow$ $\hat{\delta}_N(q_0^N, \omega) \cap F_N \neq \emptyset$

$\gamma \in \{Q' \subseteq Q : Q' \cap F_N \neq \phi\}$

$\gamma \subseteq Q,$

$\gamma \cap F_N \neq \phi$




for the subset construction

Need to prove a Lemma:

**Lemma:** $\forall w, \forall q \in Q_N, \ \hat{\delta}_D(\underbrace{\{q\}}_{\text{a state of DFA}}, w) = \hat{\delta}_N(\underbrace{q}_{\text{a state of NFA}}, w)$ = set of states of NFA subset of $Q$

Induction of length of w.

Base case: $|w| = 0 \Rightarrow w = \varepsilon$

LHS: $\hat{\delta}_D(\{q\}, \varepsilon) = \{q\}$

RHS: $\hat{\delta}_N(q, \varepsilon) = \{q\}$

$\therefore$ Base case holds

I.S.

state of DFA

LHS: $\hat{\delta}_D(\{q\}, xa) = \delta_D(\hat{\delta}_D(\{q\}, x), a)$

state of NFA

RHS: $\hat{\delta}_N(q, xa) = \bigcup_{r \in \hat{\delta}_N(q,x)} \delta(r, a)$

Show match.

IH.

$\delta_D(\boxed{\hat{\delta}_N(q, x)}, a)$ $Q'$

$= \bigcup_{q \in \hat{\delta}_N(q,x)} \delta_N(q, a)$ used defn of $\delta_D$

$\delta^D(Q' \subseteq Q, a) = \bigcup_{q \in Q'} \delta_N(q, a)$

state of NFA

$\hat{\delta}_N(q, \varepsilon) = \{q\}$

$\hat{\delta}_N(q, x \cdot a)$

$= \bigcup_{r \in \hat{\delta}(q,x)} \delta(r, a)$

state of DFA

$\hat{\delta}_D(q, \varepsilon) = q$

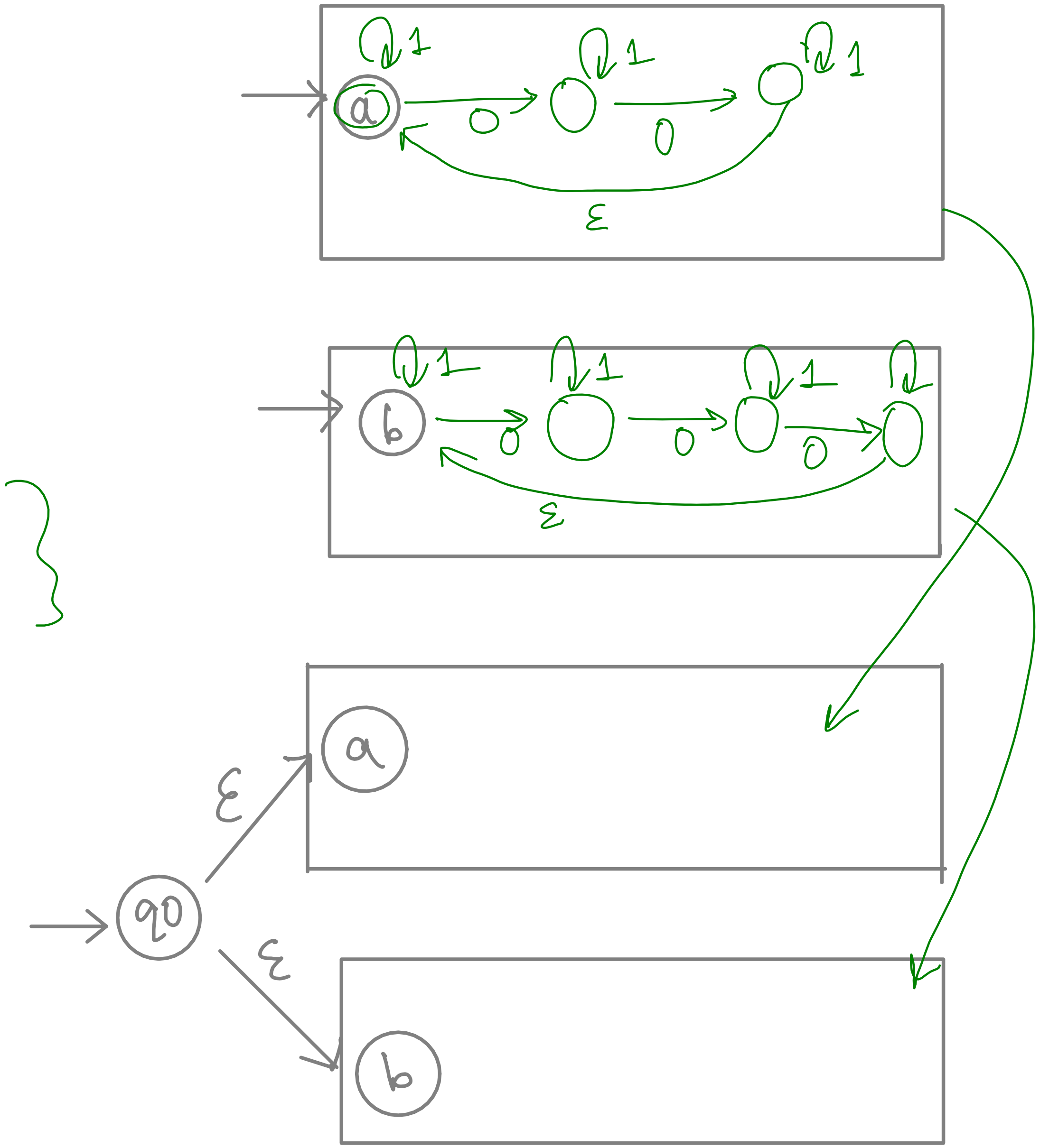
$\hat{\delta}_D(q, xa)$

$= \delta(\hat{\delta}(q, x), a)$

# Example

L = sequences of 0s of length either divisible by 2 or divisible by 3 (or both)

$K = \{ w : w \text{ has } \dots\}$



$$\text{DFA} = \text{NFA} = \varepsilon\text{-NFA}$$

= regular language

languages that are accepted by any DFA/NFA/ $\varepsilon$-NFA

REGULAR = class of regular languages

Class / complexity class = set of languages

$= \{ L : \exists \text{DFA } D \text{ s.t. } L = L(D) \}$