CSE322 Theory of Computation(L3,4)

Recap of last lecture

https://automatonsimulator.com/

Today

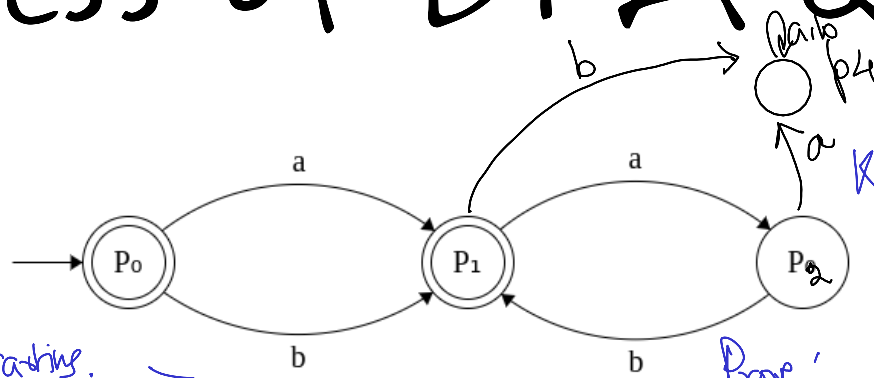
Correctness of DFA

Correctness of NFA

(extended tr. function is not to be used for HQ1, HQ2)

Correctness of DFA & NFA

bbbbbbb ×



$K = \{\varepsilon, a, b, aab, aabab, aababab, \ldots\}$
babab …

D
$= \langle \ldots,$ starting state $= p_0 \ldots \rangle$

Prove: $aab \in L(D) \equiv D$ accepts aab

$K = \{ e \} \cup \{ w :$ w starts with a or b, and is followed by zero or more sequences of the string 'ab' $\}$

set of strings that D accepts

Is $L(D) = K$?

a) $L(D) \subseteq K$

$\forall x \in L(D)$,
$x \in K$

b) $K \subseteq L(D)$

$\forall y \in K$, $y \in L(D)$

$|aab| = 3$

$\exists\, r_0\, r_1\, r_2\, r_3$ s.t.

(1) (2) (3)

Let $r_0 = p_0$, $r_1 = p_1$,
$r_2 = p_2$, $r_3 = p_1$

(1) $r_0 = p_0$ ✓

(2) $r_3 = p_1 \in F$ ✓

(3) $\delta(r_0 = p_0, a) = r_1 = p_1$ ✓

$\delta(r_1 = p_1, a) = r_2 = p_2$ ✓

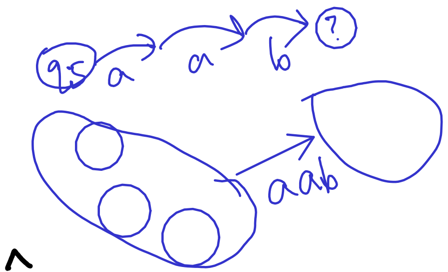
$\delta(r_2 = p_2, b) = r_3 = p_1$ ✓

DFA

Extended-delta / transition fn.

$\hat{\delta}: Q \times \Sigma^* \to Q$

Sometimes $\hat{\delta}$ will be written as δ'



$\hat{\delta}(q, \epsilon) = q$

concatenation

$q \overset{x}{\rightsquigarrow} q' \xrightarrow{a} q''$

$$\hat{\delta}(q, \underbrace{x \cdot a}_{y=}) = \delta(\underbrace{\hat{\delta}(q, x)}_{q'}, a) \qquad x \in \Sigma^*, \; a \in \Sigma$$

~~$\hat{\delta}(Q', w)$: action of FA starting from any state in Q' & string w~~

~~$\hat{\delta}(Q', w)$~~ $= \bigcup_{q \in Q'} \{\hat{\delta}(q, w)\}$

$Q' \subseteq Q$

Q: Redefine "M accepts x" using extended trans. fn.

Claim:- M accepts x iff $\hat{\delta}(q_0, x) \in F$

$= \langle \ldots, q_0 \ldots \rangle$

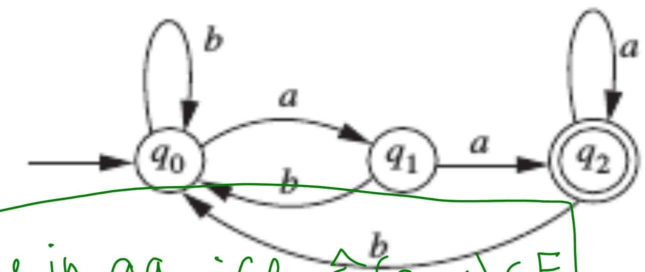
Proof:- Proof by induction on length of x. (Exercise)

When $|x| = 0 \Rightarrow x = \varepsilon$

M accepts ε iff $\underbrace{\hat{\delta}(q_0, \varepsilon)}_{q_0} \in F$

$\Leftrightarrow$ M accepts ε iff $q_0 \in F$ (already holds)

Proof of correctness



x ends in aa iff $\hat{\delta}(q_0, x) \in F$

$L = \{ x : x \text{ is a binary string that ends with aa} \}$

Prove that : $L(D) = L \iff x \in L(D)$ iff $x \in L$. $\iff$ $\boxed{x \in L \text{ iff } \hat{\delta}(q_0, x) \in F}$

$x = y \cdot a$

$\hat{\delta}(q_0, y)$

a) if $x \in L$, then $\hat{\delta}(q_0, x) \in F$

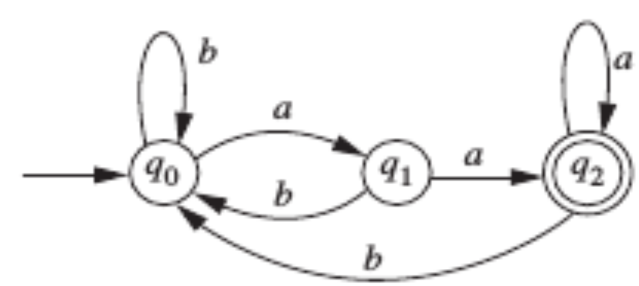
b) if $\hat{\delta}(q_0, x) \in F$ then $x \in L$.

What can we say about "computations" ending in q0? $q_0 \overset{?}{\rightsquigarrow} q_0$

What can we say about "computations" ending in q1? $q_0 \overset{?}{\rightsquigarrow} q_1$

What can we say about "computations" ending in q2? $q_0 \rightsquigarrow q_2$ $\overset{?}{\longrightarrow} L$

Proof of correctness



L1 = { x : x is a binary string that ends with aa }

Prove that L(N) = L1

Level-1: By definition, w is in L1 iff $q2 = \delta'(q0,w)$. We will do induction on the length of w to prove the following 3 facts.

(1) w=e or w ends in b iff $\delta'(q0,w)=q0$

(2) w=a or w ends in ba iff $\delta'(q0,w)=q1$

(3) $\boxed{\text{w ends in aa iff } \delta'(q0,w)=q2}$

(1) ($\varepsilon=\varepsilon$ or ε ends in b) iff ($\hat{\delta}(q0,\varepsilon)=q0$)
True ✓ $=q0$

(2) ($\varepsilon=a$ or ε ends in ba) iff ($\hat{\delta}(q0,\varepsilon)=q1$)
False ✓ False

(3) ε ends in aa iff $\hat{\delta}(q0,\varepsilon)=q2$
False False

Level-2:

(Base case) |w|=0, i.e., w=e. (Prove all three iff statements)

Induction hypothesis: All three facts are true for any string of length <= n.

Induction step: To show that all three facts are true for any w of length n+1.

Let $w = x \cdot s$ where $s \in \Sigma$, $x \in \Sigma^*$ and $|x| = n$

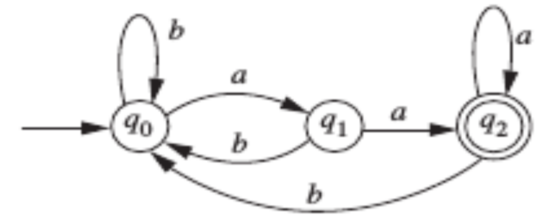
Proving (1) forward direction

If $xs = \varepsilon$ or xs ends in b then $\hat{\delta}(q0, xs) = q0$

Let $w \neq \varepsilon$ or w ends in b. $\rightarrow s = b$ To show $\delta'(q0, xb) = q0 = \delta(\underbrace{\hat{\delta}(q0, x)}_{q'}, b) = \delta(q', b) = q0$

$\because |w| \geq 1$

$\because \hat{\delta}(q, b) = q0 \quad \forall q \in Q$ $\quad \therefore \delta'(q0, x \cdot b) = \delta'(\delta'(q0, x), b) = q0$



Proving (1) backward direction

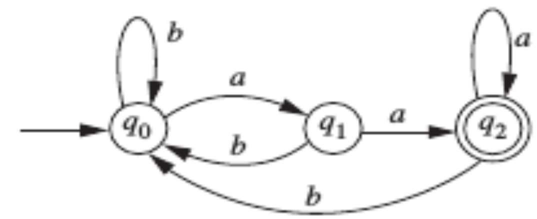
Let $\delta'(q0, xs) = q0$. To show $w \neq \varepsilon$ or w ends in b. $\rightarrow s = b$

$q0 = \delta(\underbrace{\delta'(q0, x)}_{q'}, s)$ No state q' s.t. $\delta(q', a) = q0$ $\therefore s \neq a$

Suppose $s = a$. But $\forall q \in Q$, $\delta'(q, a) \neq q0$ $\therefore \delta'(\delta'(q0, x), s) \neq q0$.

On the other hand, if $s = b$, then $\forall q \in Q$, $\delta'(q, s) = q0$

$\therefore s = b$



Proving (3) forward direction

$|ya| = n$

Let w end in aa, i.e., $w = xa = yaa$ where $|y| = n-1$. To show $\delta'(q0, yaa) = q2$

ya can be of three types

- $ya = a$. Then, $\delta'(q0, ya) = \delta'(q0, a) = q1$
- ya ends in ba. Then $\delta'(q0, ya) = q1$ (by IH)

$\Big\}$ $\delta'(q0, yaa) = \delta'(q1, a) = q2$

- ya ends in aa. Then $\delta'(q0, ya) = q2$ (by IH) $\Rightarrow \delta'(q0, yaa) = \delta'(q2, a) = q2$

Proving (3) backward direction

Let $\delta'(q0, xs) = q2$. To show xs ends in aa.

x ends in a and $s = a$.

So, $\delta(\underbrace{\delta'(q0, x)}_{?}, \underbrace{s}_{?}) = q2$

$\because \delta(q, b) \neq q2 \ \forall q, \ \delta(q0, a) \neq q2, \ \delta(q1, a) = q2, \ \delta(q2, a) = q2$ $\Big\}$ $\delta'(q0, x) \in \{q1, q2\}$, $s = a$

Case $\delta'(q0, x) = q1$. Since $|x| = n$, by IH, $x = a$ or x ends in ba.

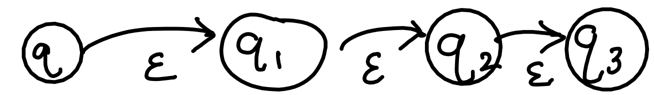
Case $\delta'(q0, x) = q2$. Since $|x| = n$, by IH, x ends in aa.

$\Big\}$ In all cases, xa ends in aa which proves the IS for (3).

(2) Exercise.

Extended transition function for NFA

$\hat{\delta}(q,\varepsilon) = ?$ $\{q, q^1, q^2, q^3\} = E(q)$



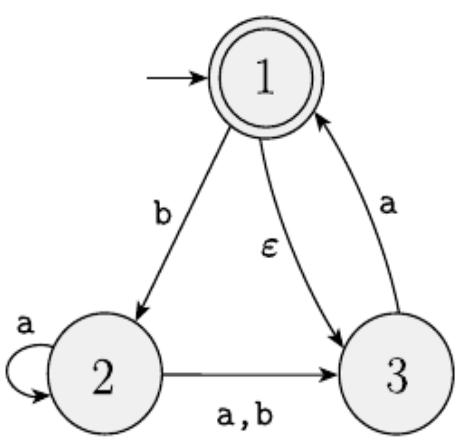
ε-closure

For any set $R \subseteq Q$, $E(R)$ is defined recursively as:

(1) $R \subseteq E(R)$

(2) $\forall q \in E(R)$, $\underbrace{\boxed{\delta(q,\varepsilon)}}_{\text{NFA}} \subseteq E(R)$

Algo: Initialize $E(R)=R$
Keep applying (2) until $E(R)$ changes



$E(1) = \{1, 3\}$

$E(2) = \{2\}$

$E(3) = \{3\}$

$E(1,3) = \{1,3\}$

$E(2,3) = \{2,3\}$

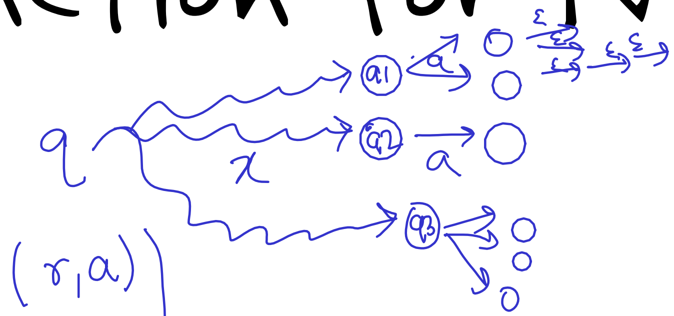
$E(q) = \{ q' \mid q \xrightarrow{\text{0 or more } \varepsilon\text{-rules}} q' \}$

$E(\{q1, q2, \ldots, qk\}) = ?$

$\left.\begin{array}{l} E(1,2) = \\ E(1,2,3) = \end{array}\right\}$ Exercise

$E(\{\}) = \{\}$

Extended transition function for NFA



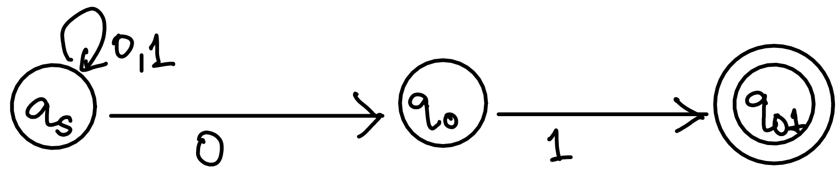
$\hat{\delta}(q, \varepsilon) = ?$ $\quad E(\{q\})$

$$\hat{\delta}(q, x \cdot a) = E\left(\bigcup_{r \in \hat{\delta}(q, x)} \delta(r, a) \right) \qquad E\left(\bigcup_{\boxed{r \in \hat{\delta}(q,x)}} \delta(r, a) \right) \quad \text{for } x \in \Sigma^*, \ a \in \Sigma$$

$\{q_1, q_2, q_3\}$ $\qquad r \in E(\ldots\ldots)$

Claim:- An NFA $N = \langle Q, \Sigma, \delta, q_0, F \rangle$ accepts a string x if $\hat{\delta}(q_0, x) \cap F \neq \emptyset$

Proof of correctness

N: 

$L1 = \{ w \text{ endin with } 01 \}$

Prove that $L(N) = L1$.

N accepts ω iff $\omega \in L1$

$\hat{\delta}(q_s, w) \ni q_{01}$ iff w ends in 01

Proof level-1: By definition, $\omega \in L1$ iff $q_{01} \in \hat{\delta}(q_s, \omega)$. We will use induction on the length of ω to prove the following 3 facts:

(a) ω ends in 01 iff $q_{01} \in \hat{\delta}(q_s, \omega)$

(b) ω ends in 0 iff $q_0 \in \hat{\delta}(q_s, \omega)$

(c) For all ω, $q_s \in \hat{\delta}(q_s, \omega)$

Difficult to show correct by only proving (a).

Proof level-2:

Base case: $|w| = 0$

Induction step: $w = xa$, for $|x| = n$ and a is a symbol.

Assume that induction claims are valid for x.

...

Assume that induction claims are valid for x.

That is,

1. x ends in 01 iff $\hat{\delta}(q_s, x) \ni q_{01}$
2. x ends in 0 iff $\hat{\delta}(q_s, x) \ni q_0$
3. For all x, $\hat{\delta}(q_s, x) \ni q_s$

To show that

1. xa ends in 01 iff $\hat{\delta}(q_s, xa) \ni q_{01}$
2. xa ends in 0 iff $\hat{\delta}(q_s, xa) \ni q_0$
3. For all x, a $\hat{\delta}(q_s, xa) \ni q_s$

?