Boolean Functions: 1,2,3 and more

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Outline

What is a Boolean function?

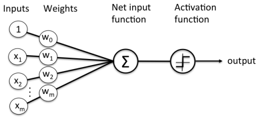
Mapping from $\{0,1\}^* \to \{0,1\}^*$

Can be used to model (almost) any mapping (over discrete domain and range)

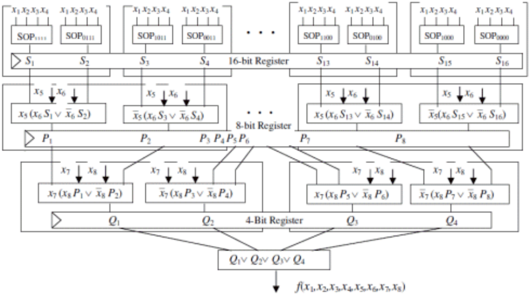
This talk is about $m = 2^n$-bit to 1-bit Boolean functions.

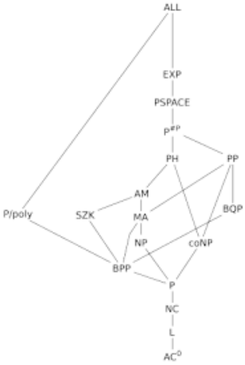
x	$f_1(x)$	$f_2(x)$
000	1101	0
001	0001	0
010	1101	0
$\dots$	$\dots$	$\dots$

Where to find them?



Schematic of Rosenblatt's perceptron.





Examples

Consider m-to-1 bit $f()$.
$x = x_1 \ldots x_m$ and $\|x\| = m$ (number of bits).
$|x|$ denotes number of ones in x.

Total functions

$OR(x) = 1$ if any bit of x is 1, $= 0$ otherwise
$AND(x) = 1$ if all bits of x are 1, $= 0$ otherwise
$PARITY(x) = 1$ if odd number of bits of x are 1, $= 0$ otherwise
$MAJORITY(x) = 1$ if more than $\frac{m}{2}$ bits of x are 1
$Threshold_k(x) = 1$ if at least k bits of x are 1

$INDEX(x) = 1$ if x can be written as $x = y \cdot z$ *s.t.*

- $\|y\| = \lfloor \log_2(\|z\|) \rfloor$
- $z_{int(y)} = 1$

Example: $INDEX(10 \cdot 1001) = 0$ and $INDEX(11 \cdot 0101) = 1$

Examples

Promise problems: Domain of f is strict subset of $\{0,1\}^*$

Partial functions

DJ: Given that $|x| \in \{0, \frac{n}{2}, n\}$, $DJ(x) = 0$ if $|x| = \frac{n}{2}$

EWDP: Given that $|x| \in \{k_l, k_u\}$, $EWDP_{k_l,k_u}(x) = 1$ if $|x| = k_u$

WDP: Given that $|x| \leq k_l$ or $|x| \geq k_u$, $EWDP_{k_l,k_u}(x) = 1$ if $|x| \geq k_u$

Function problems

Consider inputs of length 2^n.

$$x = \langle g(\overbrace{00\dots0}^{n}), g(00\dots1), \dots, g(11\dots1)\rangle$$

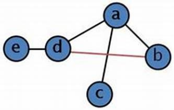
Querying x_j is same as querying g(bit repr. of j)
e.g. $x_5 \equiv g(0101)$

Given that $g(y) = a \cdot y \oplus b$, output a, b.

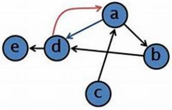
Given $b \in \{0,1\}^n$, $\rho \in [0, 2^n]$, determine if $|\sum_y (-1)^{g(y)\oplus b\cdot y}| \geq \rho$

y	00	01	10	11
$g(y)$	0	1	1	0
$(-1)^{g(y)}$	1	-1	-1	1
$(-1)^{g(y)\oplus(01)\cdot y}$	1	1	-1	-1

Push the pedal



	a	b	c	d	e
a	0	1	1	1	0
b	1	0	0	1	0
c	1	0	0	0	0
d	1	1	0	0	1
e	0	0	0	1	0



	a	b	c	d	e
a	0	1	0	1	0
b	0	0	0	1	0
c	1	0	0	0	0
d	1	0	0	0	1
e	0	0	0	0	0

Input $g(y)$ denotes the "edge" function of a graph G

Suppose $g(y)$ denotes the "edge" function of a graph G.

$g(y_1 \dots y_{\frac{n}{2}} \;\cdot\; y_{\frac{n}{2}+1} \dots y_n)$: is there an edge between vertex $y_1 \dots y_{\frac{n}{2}}$ and vertex $y_{\frac{n}{2}+1} \dots y_n$ of a G with $2^{n/2}$ vertices?

Does G have a Hamiltonian circuit?

Outline

Decision tree complexity

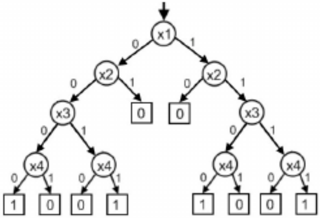
Input: m-bit $\{0,1\}^*$-string x, or
n-bit Boolean function $g : \{0,1\}^n \to \{0,1\}$ (denote 2^n by m)

Query Complexity

= largest number of bits/$g()$ queried on any input

Relevant when querying is "costlier" than local operations.

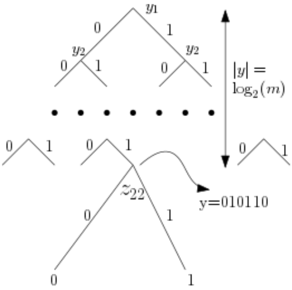
Trivially, query complexity is at most m.



Decision tree for $VERIFY_4(x)$

Query complexity of $INDEX$

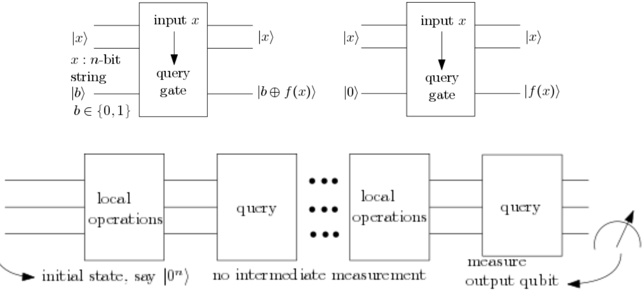
$INDEX(x) = 1$ if x can be written as $x = y \cdot z$ *s.t.*
z represents a "database" and
y is a position of database entry
yth bit of z should be 1



Query complexity $= 1 + \log_2(m)$

Quantum query complexity

Intermediate measurements can modify state.
So, measure only at end!



Error in computing Boolean functions

$$X_0 : \text{set of inputs for which } f(x) = 0$$
$$X_1 : \text{set of inputs for which } f(x) = 1$$

Bounded error algorithms

For $x \in X_0$, $\Pr[Algo(x) \to 1] \leq \rho_l < 1/2$
For $x \in X_1$, $\Pr[Algo(x) \to 0] \leq \rho_u < 1/2$

More chance of being correct than wrong.

How good (or bad) is quantum?

$D(f)$: optimal #bits queried by deterministic classical algo.
$R(f)$: optimal #bits queried by bounded-error classical ...
$Q_E(f)$: optimal #query-gates in "exact" quantum algo.
$Q_2(f)$: optimal #query-gates in bounded-error quantum ...

Questions?

For a given f, what are $D(f)$, $Q_E(f)$ and $Q_2(f)$?

Is there a general relationship between them?

Are there functions for which $Q_E(f) \ll D(f)$?

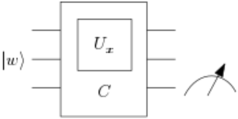
Are there functions for which $Q_2(f) \ll R(f)$?

Are there functions for which $Q_E(f) \ll R(f)$?

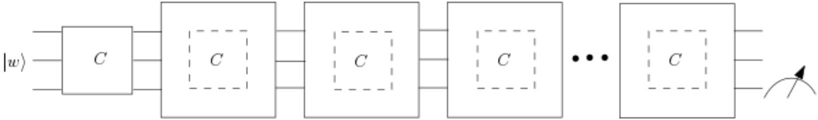
Input functions could be total or partial

Reducing algorithm error

Technique of choice: Repeat algorithm several times



$\Pr[\text{observing } |1\rangle] = sin^2\theta$



Amplitude Amplification (AA)

$\Pr[\text{observing } |1\rangle] = \sin^2[\theta + 2t\theta]$
$2t$ can be replaced by rt for any fixed $r \leq 2$

Using amplitude ampl. to reduce error

prob. of success = $1/100$

# calls to U_x	prob. of success
1	$1/100$
3	$9/100$
7	$41/100$
11	$80/100$
15	$99.5/100$

prob. of success = $1/4 = \sin^2 30$

# calls to U_x	prob. of success
1	$1/4$
2	1 !!!

prob. of success = $1/2 = \sin^2 45$

# calls to U_x	prob. of success
1	$1/2$
2	1 !!!

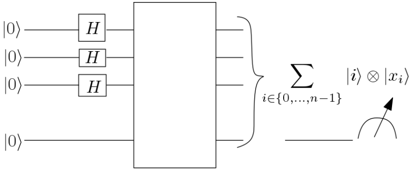
Outline

Complexity of DJ : $|x| = m/2$ or $|x| \in \{0, m\}$

Consider m-bit y *s.t.* $y_i = x_i \oplus x_0$.

$\boxed{|x| = m/2 \text{ same as } |y| = m/2 \quad vs. \quad |x| \in \{0, m/2\} \text{ same as } |y| = 0}$

Querying y_i requires two queries to x



Apply AA to increase prob. of $|1\rangle$ (Bera,2015)

If $|y| = 0$, then observe $|1\rangle$ with prob. 0.
After AA, observe $|1\rangle$ with prob. 0.

If $|y| = m/2$, then observe $|1\rangle$ with prob. 1/2.
After AA using 1 iteration, observe $|1\rangle$ with prob. 1.

Complexity of DJ

zero-error quantum algorithm using 6 query-gates

Quantum *vs.* Classical

zero-error deterministic algorithm needs $\frac{m}{2} + 1$ queries
6 query randomized algorithm has at least $1/2^{12}$ prob. of error

Deutsch-Jozsa's circuit (Deutsch-Jozsa,1992)

zero-error quantum algorithm using 1 query gate !!!

Outline

Separation for partial functions

Deutsch-Jozsa's problem (DJ)

- efficiently solvable by quantum circuit (exactly in constant queries)
- exact classical algo is inefficient (50% bits are queried)
- bounded-error classical algo is moderate (error $1/2^{2k}$ with k queries)

Simon's problem

- efficiently solvable by quantum circuit (bounded-error requires $\log m$ queries)
- bounded-error classical algo is bad (requires $\Omega(m)$ queries)

Separation for total functions

Classical is not too worse *vis-a-vis.* quantum

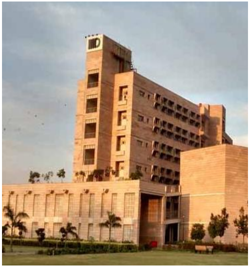
- Any q-query exact quantum algo.
  $\implies$ q^3-query exact classical algo.
- Any q-query bounded-error quantum algo.
  $\implies$ q^6-query exact classical algo.

Quantum is somewhat better than classical

- OR: bounded-error quantum makes $\sqrt{m}$ queries whereas bounded-error classical makes $m/2$ queries
- There exists an f for which bounded-error classical algo. makes nearly q^3-queries whereas a bounded-error quantum algo. makes q queries

Thank you

Questions?



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Problems to work on . . .

- Optimal quantum query complexity of combinatorial (NP-complete, . . .) problems
- Fancy uses and extensions of AA (in ML?)
- Characterize query complexity based on degree, sensitivity, etc. properties of a Boolean function
- Find total functions for which quantum algo. makes q queries and optimal bounded-error algo. makes $\geq q^4$ queries
- Quantum circuit complexity of problems